**CSC238:Discrete Mathematics for Computer Science – Spring 2003**

# Solutions for Assignment 1

1. Consider the function $f(n)$ defined by

$$f(0) = 5,$$
$$f(n) = f(n-1) + 6n - 3, \quad \text{for } n \geq 1. \qquad (1)$$

Using mathematical induction, prove that $f(n) < 10n^2$ for all natural numbers $n \geq 1$.

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**Proof.** Let $S(n)$ be the statement that $f(n) < 10n^2$.

**Base Case:** n = 1. From (1) above, it follows that $f(1) = f(0) + 6 - 3 = 5 + 3 = 8$. Since $10n^2 = 10$ for $n = 1$, it follows that $S(1)$ is true.

Let $n$ be a natural number, $n \geq 1$.

**Induction Hypothesis.** Suppose $S(n)$ is true.

**Induction Step.** We need to prove $S(n+1)$ is true.

From equation (1) it follows that

$$\begin{aligned} f(n+1) =& f(n) + 6(n+1) - 3, \quad \text{by}(1), \text{since } n+1 \geq 1, \\ <& 10n^2 + 6(n+1) - 3, \quad \text{by the IH}, \\ =& 10n^2 + 6n + 3, \quad \text{by algebra}, \\ =& 10((n+1)^2 - 2n - 1) + 6n + 3, \quad \text{by algebra}, \\ =& 10(n+1)^2 - 14n - 7, \quad \text{by algebra}, \\ <& 10(n+1)^2, \quad \text{since } n \geq 1. \end{aligned}$$

Therefore $S(n+1)$ is true.

It follows by mathematical induction that $S(n)$ is true for all natural numbers $n \geq 1$.

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2. Consider the function $f(n)$ defined by

$$\begin{aligned} f(0) &= 5, \\ f(1) &= 4, \\ f(n) &= 3f(\lfloor n/2 \rfloor) + 2^n, \quad \text{for } n \geq 2. \end{aligned} \tag{2}$$

Here $\lfloor n/2 \rfloor$ equals the largest natural number less than or equal to $n/2$. So $\lfloor 3/2 \rfloor = 1$, $\lfloor 4/2 \rfloor = 2$, and so on. Using mathematical induction, prove that $f(n) \leq 2^{n+2}$ for all $n \geq 1$.

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It is useful to first prove the following:

**Lemma 2.1.** For natural numbers $n \geq 2$ the integer $m = \lfloor (n+1)/2 \rfloor$ satisfies $1 \leq m \leq n-1$.

**Proof of Lemma 2.1.** For $n = 2$ we have $m = \lfloor (2+1)/2 \rfloor = 1$. Therefore $1 \leq m \leq n-1$ for this case.

Suppose $n \geq 3$. By the definition of floor, $m$ is an integer such that $(n+1)/2 = m + \delta$, where $\delta \in 0, 1/2$. Therefore,

$$\begin{aligned} m =& (n+1)/2 - \delta \\ \leq& (n+1)/2 \ \text{ since } \delta \geq 0, \\ \leq& (n+1)/2 + (n-3)/2 \ \text{ since } n \geq 3, \\ =& n - 1 \ \text{ by algebra }. \end{aligned}$$

A lower bound for $m$ can be obtained as follows,

$$\begin{aligned} m =& (n+1)/2 - \delta \\ \geq& (n+1)/2 - 1/2 \ \text{ since } \delta \leq 1/2, \\ =& n/2 \\ \geq& 1 \ \text{ since } n \geq 3. \end{aligned}$$

Therefore we have shown $1 \leq m \leq n-1$ for $n \geq 3$. This completes the proof of Lemma 2.1.

**Proof for Question 2.** Let $S(n)$ be the statement that $f(n) \leq 2^{n+2}$.

**Base Cases:** $n = 1, 2$. From (2) above, it follows that $f(1) = 4$, and $f(2) = 3f(1) + 2^2 = 16$. Since $2^{n+2}$ equals 8 and 16 for $n = 1$ and 2, respectively, it follows that both $S(1)$ and $S(2)$ are true.

Let $n$ be a natural number, $n \geq 2$.

**Induction Hypothesis.** Suppose $S(k)$ is true for each integer $k$ with $1 \leq k \leq n$.

**Induction Step.** We need to prove $S(n+1)$ is true.

Since $n \geq 2$ it follows that $n+1 \geq 2$ and the bottom equation in (2) implies

$$f(n+1) = 3f(\lfloor (n+1)/2 \rfloor) + 2^{n+1}.$$

Define $m = \lfloor (n+1)/2 \rfloor$. Since $n \geq 2$ we know from Lemma 2.1 that $m$ satisfies $1 \leq m \leq n-1$. Therefore, by the induction hypothesis, $S(m)$ is true. That is, $f(m) \leq 2^{m+2}$. Using this in the equation above we find

$$\begin{aligned} f(n+1) =& 3f(m) + 2^{n+1}, \text{ since } n+1 \geq 2 \\ \leq& 3(2^{m+2}) + 2^{n+1}, \text{ by the IH and } 1 \leq m \leq n-1, \\ \leq& 3(2^{n-1+2}) + 2^{n+1}, \text{ since } m \leq n-1 \\ =& 2^{n+1}(3+1) = 2^{(n+1)+2}, \text{ by algebra.} \end{aligned}$$

Therefore we have proved that $S(n+1)$ is true.

By mathematical induction it follows that $S(n)$ is true for all natural numbers $n \geq 1$.

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3. Consider the function $f(n)$ defined by

$$\begin{aligned} &f(0) = 0, \\ &f(1) = 1, \\ &f(n) = f(n-1) + f(n-2), \text{ for } n \geq 2. \end{aligned} \tag{3}$$

Using mathematical induction, prove that

$$f(n)f(n+1) = \sum_{k=0}^{n} f^2(k), \tag{4}$$

for all natural numbers $n \geq 0$. (Since we want you to practice induction, proofs which do not rely on induction will receive zero marks.)

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**Proof.** Let $S(n)$ be the statement that $f(n)f(n+1) = \sum_{k=0}^{n} f^2(k)$.

**Base Case:** For $n = 0$, it follows from (3) that $f(0)f(1) = 0 * 1 = 0$, and $f(0)^2 = 0$. Therefore $S(0)$ is true.

Let $n$ be a natural number, $n \geq 1$.

**Induction Hypothesis.** Suppose $S(n-1)$ is true.

**Induction Step.** We need to prove $S(n)$ is true.

From equation (3) it follows that

$$\begin{aligned} f(n+1)f(n) =& (f(n)+f(n-1))f(n), \text{ by (3) since } n+1 \geq 2, \\ \leq& f(n)^2 + f(n-1)f(n), \text{ by algebra}, \\ =& f(n)^2 + \sum_{k=0}^{n-1} f^2(k), \text{ by the IH, since, } n-1 \geq 0, \\ =& \sum_{k=0}^{n} f^2(k), \text{ by algebra.} \end{aligned}$$

Therefore $S(n)$ is true.

It follows by mathematical induction that $S(n)$ is true for all natural numbers $n$.

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4. The height of a non-empty tree is defined to be the maximum number of edges in any path from the root of the tree to a leaf node. For an empty tree, we define the height to be -1. Prove that, for each integer $h \geq -1$, if $n$ is the number of nodes in a full binary tree of height $h$ then $n \leq 2^{h+1} - 1$. Your proof must rely on mathematical induction.

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**Proof.** Let $S(h)$ be the statement that a full binary tree of height $h$ has at most $2^{h+1} - 1$ nodes.

**Base Case:** Consider $h = -1$. Any tree that has height $-1$ must be empty. Therefore it has $n = 0$ nodes. Note $2^{-1+1} - 1 = 1 - 1 = 0$. Therefore $S(-1)$ is true.

Let $h$ be a natural number.

**Induction Hypothesis.** Suppose $S(k)$ is true for $-1 \leq k < h$.

**Induction Step.** We need to prove $S(h)$ is true.

Let $T$ be any tree of height $h$. Since $h \geq 0$, $T$ cannot be empty. Therefore $T$ has a root node, along with (possibly empty) left and right subtrees $R$ and $L$, respectively. Since $T$ has height $h$ it follows that $R$ and $L$ must have height at most $h - 1$. Also, since $R$ and $L$ are trees their heights must each be at least $-1$. Therefore the induction hypothesis applies to both $L$ and $R$. That is, they each must have at most $2^{(h-1)+1} - 1 = 2^h - 1$ nodes. Since the nodes in $T$ just consist of the root node and any nodes in the left and right subtrees, we have the total number of nodes in $T$ must be at most $1 + 2(2^h - 1) = 2^{h+1} - 1$, as desired. Therefore, $S(h)$ is true.

It follows by mathematical induction that $S(h)$ is true for all integers $h \geq -1$.

5. Let $K_n$ denote the set of all binary strings of length $n$. That is,

$$K_n = \{a \; : \; a =< a_1, a_2, \ldots, a_n >, \text{ with } a_i = 0 \text{ or } 1 \text{ for each } i = 1 \ldots n\}.$$

Here $< a_1, a_2, \ldots, a_n >$ denotes a sequence, as defined in Chapter 0 of the course notes.

Suppose $a$ and $b$ are elements of $K_n$. Let $d(a, b)$ be the distance between $a$ and $b$, which is defined to be the number of indices $i$ at which $a_i \neq b_i$. In particular,

$$d(a, b) = |\{i \; : \; 1 \leq i \leq n \text{ and } a_i \neq b_i, \text{where } a =< a_1, a_2, \ldots, a_n >, b =< b_1, b_2, \ldots, b_n >\}|. \tag{5}$$

For each natural number $n \geq 1$, prove that there exist two sets $A_n$ and $B_n$ for which all of the following properties are satisfied:

(a) $|A_n| = |B_n| = 2^{n-1}$.

(b) $K_n = A_n \cup B_n$ and $\emptyset = A_n \cap B_n$.

(c) Any two distinct elements of $A_n$ are at least a distance of 2 apart, and similarly for any two distinct elements of $B_n$. That is, if $x, y \in A_n$ with $x \neq y$ then $d(x, y) \geq 2$. Similarly, if $x, y \in B_n$ with $x \neq y$ then $d(x, y) \geq 2$.

(d) For each element $x \in A_n$ there exists $y \in B_n$ such that $d(x, y) = 1$. (Note the choice of $y$ may depend on the $x$.) Similarly, for each element $x \in B_n$ there exists $y \in A_n$ such that $d(x, y) = 1$.

Your proof must rely on mathematical induction.

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**Proof.** Let $S(n)$ be the statement that there exist subsets $A_n$ and $B_n$ of $K_n$ which satisfy properties (a-d) in Question 5 above.

**Base Case:** Consider $n = 1$. Let $A_1 = \{< 0 >\}$ and $B_1 = \{< 1 >\}$. Then properties (a) and (b) are clearly satisfied. Since there is only one element in each of $A_1$ and $B_1$, property (c) is trivially satisfied. Property (d) follows since the distance $d(< 0 >, < 1 >)$ $= d(< 1 >, < 0 >) = 1$.

Let $n$ be a natural number, $n \geq 1$.

**Induction Hypothesis.** Suppose $S(n)$ is true.

**Induction Step.** We need to prove $S(n + 1)$ is true.

Let $A_n$ and $B_n$ be any two sets that satisfy properties (a-d) in Question 5. The induction hypothesis guarantees that such a pair must exist. Define

$$\begin{aligned}
A_{n+1} =& \{s : s =< a_1, a_2, \ldots, a_n, 0 > \quad \text{for} \quad a =< a_1, a_2, \ldots, a_n > \in A_n\} \cup \\
& \{s : s =< b_1, b_2, \ldots, b_n, 1 > \quad \text{for} \quad b =< b_1, b_2, \ldots, b_n > \in B_n\}, \\
B_{n+1} =& \{s : s =< a_1, a_2, \ldots, a_n, 1 > \quad \text{for} \quad a =< a_1, a_2, \ldots, a_n > \in A_n\} \cup \\
& \{s : s =< b_1, b_2, \ldots, b_n, 0 > \quad \text{for} \quad b =< b_1, b_2, \ldots, b_n > \in B_n\}.
\end{aligned}$$

We need to show that these sets $A_{n+1}$ and $B_{n+1}$ satisfy properties (a-d).

Given $x =< x_1, x_2, \ldots, x_n >$ we will use the shorthand $< x, 0 >$ to denote the sequence $< x_1, x_2, \ldots, x_n, 0 >$ , and similarly for $< x, 1 >$.

Note that if $x \in A_{n+1}$ then either $x =< a, 0 >$ for some $a \in A_n$, or $x =< b, 1 >$ for some $b \in B_n$. The number of different elements of the form $< a, 0 >$ is exactly the number of different $a$'s, that is $|A_n|$. Since $A_n$ satisfies property (a), $|A_n| = 2^{n-1}$. Similarly, the number of distinct elements of the form $< b, 1 >$ is also $2^{n-1}$. Since any elements of the form $< a, 0 >$ and $< b, 1 >$ differ (at least in the last place), the total number of elements in $A_{n+1}$ is $|A_{n+1}| = 2^{n-1} + 2^{n-1} = 2^n$. Similarly, it can be shown that $|B_{n+1}| = 2^n$. Therefore $A_{n+1}$ and $B_{n+1}$ satisfy property (a).

Next we wish to show $\emptyset = A_{n+1} \cap B_{n+1}$. Let $x \in A_{n+1}$ and $y \in B_{n+1}$, and suppose $x = y$. Then the last element of $x$ and $y$ must be equal, that is it must be either a 0 or a 1. Suppose 1 is that last digit in $x$ and $y$. By construction of $A_{n+1}$ and $B_{n+1}$ we must then have $x =< b, 1 >$ for some $b \in B_n$, and $y =< a, 1 >$ for some $a \in A_n$. But since $x = y$ we must have $a = b$. This implies that $a \in A_n \cap B_n$, contradicting property (b) for $A_n$ and $B_n$. Therefore 1 cannot be the last element in $x$ and $y$. The case in which 0 is the last element of $x$ and $y$ is similar, and also leads to a contradiction. Therefore, it must be the case that $\emptyset = A_{n+1} \cap B_{n+1}$.

To complete the proof of property (b) for $A_{n+1}$ and $B_{n+1}$ we need to show $K_{n+1} = A_{n+1} \cup B_{n+1}$. By construction, every element of $A_{n+1}$ and $B_{n+1}$ is a binary sequence of length $n+1$, and therefore $A_{n+1} \cup B_{n+1} \subseteq K_{n+1}$. To show the reverse, let $x \in K_{n+1}$. Then, $x$ must end in either 0 or 1, that is, there must be a $y \in K_n$ such that $x =< y, 0 >$ or $x =< y, 1 >$. Suppose $x =< y, 0 >$. By the choice of $A_n$ and $B_n$, we have $K_n = A_n \cup B_n$, and therefore $y \in A_n \cup B_n$. There are two cases, either $y \in A_n$ or $y \in B_n$. In either case it follows from the construction of $A_{n+1}$ and $B_{n+1}$ that $x =< y, 0 > \in A_{n+1} \cup B_{n+1}$. A similar argument shows that if $x =< y, 1 >$ then $x$ must be an element of $A_{n+1} \cup B_{n+1}$. Since these are the only two cases for $x$, and $x$ was an arbitrary element of $K_{n+1}$, it follows that $K_{n+1} \subseteq A_{n+1} \cup B_{n+1}$, as desired. Therefore $A_{n+1}$ and $B_{n+1}$ must satisfy property (b).

Consider property (c) next. Let $x, y \in A_{n+1}$ and suppose $x \neq y$. We need to show that $d(x, y) \geq 2$. By the construction of $A_{n+1}$, $x =< a, 0 >$ or $x =< b, 1 >$, and $y =< c, 0 >$ or $y =< e, 1 >$, where $a, c \in A_n$ and $b, e \in B_n$. Therefore there are four cases for $x$ and $y$.

Suppose $x =< a, 0 >$ and $y =< c, 0 >$ with $a, c \in A_n$. Since $x \neq y$ it must be the case that $a \neq c$. Therefore by property (c) for $A_n$, $d(a,c) \geq 2$. By the definition of distance, it follows that $d(< a, 0 >, < c, 0 >) = d(a,c)$ and therefore $d(x,y) \geq 2$. A similar argument applies to $x =< b, 1 >$ and $y =< e, 1 >$, with $b, e \in B_n$, showing $d(x,y) \geq 2$ holds in this case too.

Suppose $x =< a, 0 >$ and $y =< e, 1 >$ with $a \in A_n$ and $e \in B_n$. By the definition of distance, it follows that $d(< a, 0 >, < e, 1 >) = 1 + d(a,e)$. But, since $A_n$ and $B_n$ satisfy property (d), and $a \in A_n$, $e \in B_n$, we have $d(a,e) \geq 1$. Therefore $d(x,y) \geq 2$ in this case. A similar argument applies to $x =< b, 1 >$ and $y =< c, 0 >$, with $b \in B_n$ and $c \in A_n$, showing $d(x,y) \geq 2$ holds in this case too.

Since these are all the possible cases for $x, y \in A_{n+1}$, it follows that $d(x,y) \geq 2$ for all distinct elements $x$ and $y$ in $A_{n+1}$, as required. A similar argument applies to $B_{n+1}$. Therefore property (c) must hold for $A_{n+1}$ and $B_{n+1}$.

Finally, we are left with property (d). Let $x \in A_{n+1}$. Then by the construction of $A_{n+1}$ there are two cases, namely $x =< a, 0 >$ or $x =< b, 1 >$ for some element $a \in A_n$ or some $b \in B_n$. Suppose, $x =< a, 0 >$ with $a \in A_n$. Let $y =< a, 1 >$. Then by construction $y \in B_{n+1}$. Moreover $d(x,y) = d(< a, 0 >, < a, 1 >) = 1 + d(a,a) = 1 + 0 = 1$. A similar argument applies to the case $x =< b, 1 >$ with $b \in B_n$, showing that $y =< b, 0 > \in B_{n+1}$ satisfies $d(x,y) = 1$. Therefore we have shown that for any $x \in A_{n+1}$ there exists a $y \in B_{n+1}$ such that $d(x,y) = 1$. A similar argument shows that the roles of $A_{n+1}$ and $B_{n+1}$ can also be reversed. Therefore we have proven that $A_{n+1}$ and $B_{n+1}$ satisfy property (d).

Therefore $A_{n+1}$ and $B_{n+1}$ satisfy all properties (a-d), and hence $S(n+1)$ is true.

It follows by mathematical induction that $S(n)$ is true for all integers $n \geq 1$.

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6. Consider the set of binary strings of length $n$, namely $K_n$, along with the distance function $d(x,y)$, as defined in problem 5. Define the parity function $p : K_n \to \{even, odd\}$ to have the value $p(x) = even$ when the binary string $x$ has an even number of 1's, and $p(x) = odd$ otherwise. We say $x$ has even (or odd) parity if and only if $p(x) = even$ (or $p(x) = odd$, respectively).

   Let $n \geq 1$ be a natural number. Suppose $A \subseteq K_n$ such that, for any $x, y \in A$, either $x = y$ or $d(x,y) \geq 2$. Prove that both of the following statements are true:

   (a) $|A| \leq 2^{n-1}$ ,

   (b) If $|A| = 2^{n-1}$ then the elements in $A$ all have the same parity, that is, they are all even parity or all odd parity.

   For a change, you don't need to use mathematical induction for this question.

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The proof of both parts will rely on the properties of a graph formed using elements of $K_n$ as the nodes. Two nodes in the graph, say corresponding to elements $x, y \in K_n$, are connected by an edge if and only if $x$ and $y$ differ in exactly one place, that is, $d(x,y) = 1$. An important property of this graph is given in Lemma 6.1 below.

**Lemma 6.1.** Every node in this graph defined on $K_n$ is an endpoint of exactly $n$ edges.

**Proof of Lemma 6.1.** Consider any node in the graph, that is, any $x =< x_1, x_2, \ldots, x_n >\in K_n$. Define the set $N(x)$ to be

$$N(x) = \{y : y =< x_1, \ldots, x_{i-1}, \bar{x}_i, x_{i+1}, \ldots, x_n >, \\ \text{where } 1 \le i \le n, \quad \text{and } \bar{x}_i \ne x_i\}.$$

Here $\bar{x}_i$ denotes the complement of the bit $x_1$, that is if $x_i = 0$ then $\bar{x}_i = 1$ and vice versa. For each $y \in N(x)$ it follows from the definition of distance that $d(x,y) = 1$. So there is an edge in the graph between $x$ and $y$.

Moreover, by the definition of distance, if $y =< y_1, y_2, \ldots, y_n >\in K_n$ is such that $d(x,y) = 1$ then there must be exactly one index $i$ at which $x_i \ne y_i$. Since $x_i, y_i \in \{0,1\}$ it follows that $y_i = \bar{x}_i$. Therefore $y \in N(x)$. Therefore $N(x)$ is the set of *all* elements in $K_n$ that are a distance 1 away from $x$. These elements are all the nodes connected to $x$ by an edge in the graph. We refer to $N(x)$ as the set of neighbours of $x$.

The number of edges terminating at $x$ is therefore $|N(x)|$. And from the construction of $N(x)$ we have $|N(x)| = n$. Since this is true for an arbitrary $x \in K_n$, the lemma follows.

**Proof of part a.** We will prove this by contradiction. Suppose $A \subseteq K_n$ with the properties that $|A| > 2^{n-1}$ and, for every two distinct elements $x, y \in A$, $d(x,y) \ge 2$. Let $B = K_n - A$ be the complementary set to $A$. Then $|B| = |K_n| - |A| < 2^n - 2^{n-1} = 2^{n-1}$. (Here we have used $|K_n| = 2^n$, which follows from problem 5.)

Let $x \in A$ and consider the set of neighbours $N(x)$. By definition of $N(x)$, if $y \in N(x)$ then $d(x,y) = 1$ and $x \ne y$. Therefore $y$ cannot be an element of $A$. Thus $N(x) \subseteq B$ for each $x \in A$.

Let $m$ equal the number of edges in the graph on $K_n$ between elements in $A$ and elements in $B$. We have shown above that each element $x \in A$ has exactly $n$ distinct neighbours in $B$. Thus $m = |A|n > n2^{n-1}$.

But, by Lemma 6.1, each element of $B$ has exactly $n$ neighbours and thus $m$, the total number of edges between $A$ and $B$, must be bounded by $m \le |B|n$. From above we know $|B| < 2^{n-1}$. Therefore we have $m < n2^{n-1}$, contradicting the inequality $m > n2^{n-1}$ derived previously.

Therefore $|A| \le 2^{n-1}$, proving part a.

For part b we will use the following lemma.

**Lemma 6.2.** Let $x \in K_n$ and $y \in N(x)$. Then the parities $p(x)$ and $p(y)$ are different (i.e. one is even and the other is odd).

**Proof of Lemma 6.2.** By definition of $N(x)$, $y$ must be the same sequence as $x$ but with one bit changed. That is, for some $i \in \{1, 2, \ldots, n\}$, $x =< x_1, \ldots, x_n >$ and $y =< y_1, \ldots, y_n >$ with $y_k = x_k$ for $k \neq i$ and $y_i = \bar{x}_i$. If $x_i = 0$ then $\bar{x}_i = 1$ and $y$ has exactly one more bit equal to one than $x$ does. Therefore the parity of $x$ and $y$ must be different. Similarly, if $x_i = 1$ then $y_i = \bar{x}_i = 0$ and $y$ has exactly one fewer bits equal to one than $x$ does. Again the parity must be different. Since 0 and 1 are the only two possible values for $x_i$, the result follows.

**Proof of part b.** Suppose $A \subseteq K_n$ with the properties that $|A| = 2^{n-1}$ and, for every two distinct elements $x, y \in A$, $d(x, y) \geq 2$. Define $B = K_n - A$ to be the complementary set to $A$. Moreover, define

$$\begin{aligned}
A_e =& \{x : x \in A, \text{ and } p(x) = even\},\\
A_o =& \{x : x \in A, \text{ and } p(x) = odd\},\\
B_e =& \{x : x \in B, \text{ and } p(x) = even\},\\
B_o =& \{x : x \in B, \text{ and } p(x) = odd\}.
\end{aligned}$$

By the definitions of $A, B$ and parity, it follows that these four sets are all disjoint and their union is $K_n$. In particular, since $|K_n| = 2^n$ (see problem 5), $|B| = 2^n - 2^{n-1} = 2^{n-1}$. Therefore we have

$$|A| = |A_e| + |A_o| = 2^{n-1}, \tag{6}$$

$$|B| = |B_e| + |B_o| = 2^{n-1}. \tag{7}$$

Let $x \in A_e$ and consider the set of neighbours $N(x)$. By the definition of $A$, any other element in $A$ must have a distance of at least 2 from $x$, and therefore $N(x) \cap A = \emptyset$. Thus $N(x) \subseteq B$. Moreover, by Lemman 6.2, since $x \in A_e$ has even parity, $y \in N(x)$ must have odd parity.

For any set $S \subseteq K_n$ we define

$$N(S) = \{y : y \in K_n \text{ such that there exists an } x \in S \; with \; d(x, y) = 1\}.$$

Thus we have shown above that $N(A_e) \subseteq B_o$. Similarly, we can show that $N(A_o) \subseteq B_e$.

Since each element of $A_e$ has $n$ neighbours in $B_o$, there are $|A_e|n$ edges between $A_e$ and $B_o$. Since each element in $K_n$ has at most $n$ neighbours (by Lemma 6.1), it follows that there are no more than $|B_o|n$ edges between $A_e$ and $B_o$. Therefore $|B_o| \geq |A_e|$. A similar argument shows that $|B_e| \geq |A_o|$. However, from equations (6) and (7), $|A_e| + |A_o| = |B_e| + |B_o|$. Together with the previous inequalities we find that it must be the case that

$$|B_o| = |A_e| \text{ and } |B_e| = |A_o|.$$

Finally, since there are $|B_o|n$ edges with endpoints in $B_o$, and we know that there are $|A_e|n$ edges between $A_e$ and $B_o$ with $|A_e| = |B_o|$, it follows that *all* the edges with endpoints in $B_o$ must be between elements in $A_e$ and $B_o$. That is, there can be no edge between $B_o$ and $B_e$. Therefore $N(B_o) \subseteq A_e$. Similarly, we can show that $N(B_e) \subseteq A_o$. Together with the relations $N(A_e) \subseteq B_o$ and $N(A_o) \subseteq B_e$ proved above, we find that

$$N(A_e) = B_o,\ N(B_o) = A_e,\ N(A_o) = B_e,\ N(B_e) = A_o. \tag{8}$$

We need to prove that either $A_e$ or $A_o$ is empty. We will do this by contradiction.

Suppose $A_e$ and $A_o$ are both non-empty. Let $x =< x_1, \ldots, x_n >\in A_e$ and $y =< y_1, \ldots, y_n >\in A_o$. Consider the sequence $< e^0, e^1, \ldots, e^n >$ with $e^0 = x$, $e^n = y$, and $e^k =< y_1, \ldots, y_k, x_{k+1}, \ldots, x_n >$ for $1 \leq k < n$. Since $K_n = A_e \cup A_o \cup B_e \cup B_o$ and these four subsets are all disjoint, we must have $e^k$ in precisely one of these four subsets of $K_n$ for each $k$. Also, for any $k \in \{1, \ldots n\}$, either $y_k \neq x_k$, in which case $e^{k-1} \neq e^k$ and $d(e^{k-1}, e^k) = 1$, or $y_k = x_k$ and $e^{k-1} = e^k$. Therefore $e^k \in N(e^{k-1}) \cup \{e^{k-1}\}$ for all $k \in \{1, \ldots n\}$.

Define $L = \{k : e^k \in A_o \cup B_e\}$. Notice that $e^0 = x \in A_e$ implies $0 \notin L$, and $e^n = y \in A_o$ implies $n \in L$. Therefore $L$ is a non-empty subset of the natural numbers. Thus it must have a minimal element $j \in L$. Since $0 \notin L$ the minimal element cannot be 0, so $j > 0$. By the definition of $L$ we then have $e^{j-1} \in A_e \cup B_o$ and $e^j \in A_o \cup B_e$. In particular $e^{j-1} \neq e^j$ and therefore $e^j \in N(e^{j-1})$.

Therefore we have shown $e^{j-1} \in A_e \cup B_o$, equation (8) holds, and $e^j \in N(e^{j-1})$. Together these imply $e^j \in N(A_e) \cup N(B_o) = A_e \cup B_o$. But, since the sets $A_e$, $A_o$, $B_e$ and $B_o$ are all disjoint, this contradicts $e^j \in A_o \cup B_e$.

Therefore one of $A_e$ or $A_o$ must be empty, completing the proof.