CSC238:Discrete Mathematics for Computer Science – Spring 2003

Solutions for Assignment 1

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1. Consider the function $f(n)$ defined by

$$f(0) = 5,$$
$$f(n) = f(n-1) + 6n - 3, \quad \text{for } n \geq 1. \tag{1}$$

Using mathematical induction, prove that $f(n) < 10n^2$ for all natural numbers $n \geq 1$.

Proof. Let $S(n)$ be the statement that $f(n) < 10n^2$.

Base Case: n = 1. From (1) above, it follows that $f(1) = f(0) + 6 - 3 = 5 + 3 = 8$. Since $10n^2 = 10$ for $n = 1$, it follows that $S(1)$ is true.

Let n be a natural number, $n \geq 1$.

Induction Hypothesis. Suppose $S(n)$ is true.

Induction Step. We need to prove $S(n+1)$ is true.

From equation (1) it follows that

$$\begin{aligned}
f(n+1) =& f(n) + 6(n+1) - 3, \quad \text{by}(1), \text{since } n+1 \geq 1, \\
<& 10n^2 + 6(n+1) - 3, \quad \text{by the IH}, \\
=& 10n^2 + 6n + 3, \quad \text{by algebra}, \\
=& 10((n+1)^2 - 2n - 1) + 6n + 3, \quad \text{by algebra}, \\
=& 10(n+1)^2 - 14n - 7, \quad \text{by algebra}, \\
<& 10(n+1)^2, \quad \text{since } n \geq 1.
\end{aligned}$$

Therefore $S(n+1)$ is true.

It follows by mathematical induction that $S(n)$ is true for all natural numbers $n \geq 1$.

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2. Consider the function $f(n)$ defined by

$$\begin{aligned} f(0) &= 5, \\ f(1) &= 4, \\ f(n) &= 3f(\lfloor n/2 \rfloor) + 2^n, \quad \text{for } n \geq 2. \end{aligned} \tag{2}$$

Here $\lfloor n/2 \rfloor$ equals the largest natural number less than or equal to $n/2$. So $\lfloor 3/2 \rfloor = 1$, $\lfloor 4/2 \rfloor = 2$, and so on. Using mathematical induction, prove that $f(n) \leq 2^{n+2}$ for all $n \geq 1$.

It is useful to first prove the following:

Lemma 2.1. For natural numbers $n \geq 2$ the integer $m = \lfloor (n+1)/2 \rfloor$ satisfies $1 \leq m \leq n - 1$.

Proof of Lemma 2.1. For $n = 2$ we have $m = \lfloor (2+1)/2 \rfloor = 1$. Therefore $1 \leq m \leq n-1$ for this case.

Suppose $n \geq 3$. By the definition of floor, m is an integer such that $(n+1)/2 = m + \delta$, where $\delta \in 0, 1/2$. Therefore,

$$\begin{aligned} m =& (n+1)/2 - \delta \\ \leq& (n+1)/2 \text{ since } \delta \geq 0, \\ \leq& (n+1)/2 + (n-3)/2 \text{ since } n \geq 3, \\ =& n - 1 \text{ by algebra }. \end{aligned}$$

A lower bound for m can be obtained as follows,

$$\begin{aligned} m =& (n+1)/2 - \delta \\ \geq& (n+1)/2 - 1/2 \text{ since } \delta \leq 1/2, \\ =& n/2 \\ \geq& 1 \text{ since } n \geq 3. \end{aligned}$$

Therefore we have shown $1 \leq m \leq n - 1$ for $n \geq 3$. This completes the proof of Lemma 2.1.

Proof for Question 2. Let $S(n)$ be the statement that $f(n) \leq 2^{n+2}$.

Base Cases: $n = 1, 2$. From (2) above, it follows that $f(1) = 4$, and $f(2) = 3f(1) + 2^2 = 16$. Since 2^{n+2} equals 8 and 16 for $n = 1$ and 2, respectively, it follows that both $S(1)$ and $S(2)$ are true.

Let n be a natural number, $n \geq 2$.

Induction Hypothesis. Suppose $S(k)$ is true for each integer k with $1 \leq k \leq n$.

Induction Step. We need to prove $S(n+1)$ is true.

Since $n \geq 2$ it follows that $n+1 \geq 2$ and the bottom equation in (2) implies

$$f(n+1) = 3f(\lfloor (n+1)/2 \rfloor) + 2^{n+1}.$$

Define $m = \lfloor (n+1)/2 \rfloor$. Since $n \geq 2$ we know from Lemma 2.1 that m satisfies $1 \leq m \leq n-1$. Therefore, by the induction hypothesis, $S(m)$ is true. That is, $f(m) \leq 2^{m+2}$. Using this in the equation above we find

$$\begin{aligned} f(n+1) =& 3f(m) + 2^{n+1}, \quad \text{since } n+1 \geq 2 \\ \leq& 3(2^{m+2}) + 2^{n+1}, \quad \text{by the IH and } \ 1 \leq m \leq n-1, \\ \leq& 3(2^{n-1+2}) + 2^{n+1}, \quad \text{since } m \leq n-1 \\ =& 2^{n+1}(3+1) = 2^{(n+1)+2}, \quad \text{by} \quad \text{algebra.} \end{aligned}$$

Therefore we have proved that $S(n+1)$ is true.

By mathematical induction it follows that $S(n)$ is true for all natural numbers $n \geq 1$.

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3. Consider the function $f(n)$ defined by

$$\begin{aligned} &f(0) = 0, \\ &f(1) = 1, \\ &f(n) = f(n-1) + f(n-2), \quad \text{for } n \geq 2. \end{aligned} \tag{3}$$

Using mathematical induction, prove that

$$f(n)f(n+1) = \sum_{k=0}^{n} f^2(k), \tag{4}$$

for all natural numbers $n \geq 0$. (Since we want you to practice induction, proofs which do not rely on induction will receive zero marks.)

Proof. Let $S(n)$ be the statement that $f(n)f(n+1) = \sum_{k=0}^{n} f^2(k)$.

Base Case: For $n = 0$, it follows from (3) that $f(0)f(1) = 0 * 1 = 0$, and $f(0)^2 = 0$. Therefore $S(0)$ is true.

Let n be a natural number, $n \geq 1$.

Induction Hypothesis. Suppose $S(n-1)$ is true.

Induction Step. We need to prove $S(n)$ is true.

From equation (3) it follows that

$$\begin{aligned} f(n+1)f(n) =&(f(n)+f(n-1))f(n), \text{ by (3) since } n+1 \geq 2, \\ \leq& f(n)^2 + f(n-1)f(n), \text{ by algebra,} \\ =& f(n)^2 + \sum_{k=0}^{n-1} f^2(k), \text{ by the IH, since, } n-1 \geq 0, \\ =& \sum_{k=0}^{n} f^2(k), \text{ by algebra.} \end{aligned}$$

Therefore $S(n)$ is true.

It follows by mathematical induction that $S(n)$ is true for all natural numbers n.

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4. The height of a non-empty tree is defined to be the maximum number of edges in any path from the root of the tree to a leaf node. For an empty tree, we define the height to be -1. Prove that, for each integer $h \geq -1$, if n is the number of nodes in a full binary tree of height h then $n \leq 2^{h+1} - 1$. Your proof must rely on mathematical induction.

Proof. Let $S(h)$ be the statement that a full binary tree of height h has at most $2^{h+1} - 1$ nodes.

Base Case: Consider $h = -1$. Any tree that has height -1 must be empty. Therefore it has $n = 0$ nodes. Note $2^{-1+1} - 1 = 1 - 1 = 0$. Therefore $S(-1)$ is true.

Let h be a natural number.

Induction Hypothesis. Suppose $S(k)$ is true for $-1 \leq k < h$.

Induction Step. We need to prove $S(h)$ is true.

Let T be any tree of height h. Since $h \geq 0$, T cannot be empty. Therefore T has a root node, along with (possibly empty) left and right subtrees R and L, respectively. Since T has height h it follows that R and L must have height at most $h - 1$. Also, since R and L are trees their heights must each be at least -1. Therefore the induction hypothesis applies to both L and R. That is, they each must have at most $2^{(h-1)+1} - 1 = 2^h - 1$ nodes. Since the nodes in T just consist of the root node and any nodes in the left and right subtrees, we have the total number of nodes in T must be at most $1 + 2(2^h - 1) = 2^{h+1} - 1$, as desired. Therefore, $S(h)$ is true.

It follows by mathematical induction that $S(h)$ is true for all integers $h \geq -1$.

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5. Let K_n denote the set of all binary strings of length n. That is,

$$K_n = \{a \ : \ a =< a_1, a_2, \ldots, a_n >, \text{ with } a_i = 0 \text{ or } 1 \text{ for each } i = 1 \ldots n\}.$$

Here $< a_1, a_2, \ldots, a_n >$ denotes a sequence, as defined in Chapter 0 of the course notes.

Suppose a and b are elements of K_n. Let $d(a, b)$ be the distance between a and b, which is defined to be the number of indices i at which $a_i \neq b_i$. In particular,

$$d(a,b) = |\{i \ : \ 1 \leq i \leq n \text{ and } a_i \neq b_i, \text{where } a =< a_1, a_2, \ldots, a_n >, b =< b_1, b_2, \ldots, b_n >\}|. \tag{5}$$

For each natural number $n \geq 1$, prove that there exist two sets A_n and B_n for which all of the following properties are satisfied:

(a) $|A_n| = |B_n| = 2^{n-1}$.

(b) $K_n = A_n \cup B_n$ and $\emptyset = A_n \cap B_n$.

(c) Any two distinct elements of A_n are at least a distance of 2 apart, and similarly for any two distinct elements of B_n. That is, if $x, y \in A_n$ with $x \neq y$ then $d(x, y) \geq 2$. Similarly, if $x, y \in B_n$ with $x \neq y$ then $d(x, y) \geq 2$.

(d) For each element $x \in A_n$ there exists $y \in B_n$ such that $d(x, y) = 1$. (Note the choice of y may depend on the x.) Similarly, for each element $x \in B_n$ there exists $y \in A_n$ such that $d(x, y) = 1$.

Your proof must rely on mathematical induction.

Proof. Let $S(n)$ be the statement that there exist subsets A_n and B_n of K_n which satisfy properties (a-d) in Question 5 above.

Base Case: Consider $n = 1$. Let $A_1 = \{< 0 >\}$ and $B_1 = \{< 1 >\}$. Then properties (a) and (b) are clearly satisfied. Since there is only one element in each of A_1 and B_1, property (c) is trivially satisfied. Property (d) follows since the distance $d(< 0 >, < 1 >)$ $= d(< 1 >, < 0 >) = 1$.

Let n be a natural number, $n \geq 1$.

Induction Hypothesis. Suppose $S(n)$ is true.

Induction Step. We need to prove $S(n + 1)$ is true.

Let A_n and B_n be any two sets that satisfy properties (a-d) in Question 5. The induction hypothesis guarantees that such a pair must exist. Define

$$\begin{aligned} A_{n+1} =& \{s : s =< a_1, a_2, \ldots, a_n, 0 > \text{ for } a =< a_1, a_2, \ldots, a_n >\in A_n\} \cup \\ & \{s : s =< b_1, b_2, \ldots, b_n, 1 > \text{ for } b =< b_1, b_2, \ldots, b_n >\in B_n\}, \\ B_{n+1} =& \{s : s =< a_1, a_2, \ldots, a_n, 1 > \text{ for } a =< a_1, a_2, \ldots, a_n >\in A_n\} \cup \\ & \{s : s =< b_1, b_2, \ldots, b_n, 0 > \text{ for } b =< b_1, b_2, \ldots, b_n >\in B_n\}. \end{aligned}$$

We need to show that these sets A_{n+1} and B_{n+1} satisfy properties (a-d).

Given $x =< x_1, x_2, \ldots, x_n >$ we will use the shorthand $< x, 0 >$ to denote the sequence $< x_1, x_2, \ldots, x_n, 0 >$, and similarly for $< x, 1 >$.

Note that if $x \in A_{n+1}$ then either $x =< a, 0 >$ for some $a \in A_n$, or $x =< b, 1 >$ for some $b \in B_n$. The number of different elements of the form $< a, 0 >$ is exactly the number of different a's, that is $|A_n|$. Since A_n satisfies property (a), $|A_n| = 2^{n-1}$. Similarly, the number of distinct elements of the form $< b, 1 >$ is also 2^{n-1}. Since any elements of the form $< a, 0 >$ and $< b, 1 >$ differ (at least in the last place), the total number of elements in A_{n+1} is $|A_{n+1}| = 2^{n-1} + 2^{n-1} = 2^n$. Similarly, it can be shown that $|B_{n+1}| = 2^n$. Therefore A_{n+1} and B_{n+1} satisfy property (a).

Next we wish to show $\emptyset = A_{n+1} \cap B_{n+1}$. Let $x \in A_{n+1}$ and $y \in B_{n+1}$, and suppose $x = y$. Then the last element of x and y must be equal, that is it must be either a 0 or a 1. Suppose 1 is that last digit in x and y. By construction of A_{n+1} and B_{n+1} we must then have $x =< b, 1 >$ for some $b \in B_n$, and $y =< a, 1 >$ for some $a \in A_n$. But since $x = y$ we must have $a = b$. This implies that $a \in A_n \cap B_n$, contradicting property (b) for A_n and B_n. Therefore 1 cannot be the last element in x and y. The case in which 0 is the last element of x and y is similar, and also leads to a contradiction. Therefore, it must be the case that $\emptyset = A_{n+1} \cap B_{n+1}$.

To complete the proof of property (b) for A_{n+1} and B_{n+1} we need to show $K_{n+1} = A_{n+1} \cup B_{n+1}$. By construction, every element of A_{n+1} and B_{n+1} is a binary sequence of length $n+1$, and therefore $A_{n+1} \cup B_{n+1} \subseteq K_{n+1}$. To show the reverse, let $x \in K_{n+1}$. Then, x must end in either 0 or 1, that is, there must be a $y \in K_n$ such that $x =< y, 0 >$ or $x =< y, 1 >$. Suppose $x =< y, 0 >$. By the choice of A_n and B_n, we have $K_n = A_n \cup B_n$, and therefore $y \in A_n \cup B_n$. There are two cases, either $y \in A_n$ or $y \in B_n$. In either case it follows from the construction of A_{n+1} and B_{n+1} that $x =< y, 0 >\in A_{n+1} \cup B_{n+1}$. A similar argument shows that if $x =< y, 1 >$ then x must be an element of $A_{n+1} \cup B_{n+1}$. Since these are the only two cases for x, and x was an arbitrary element of K_{n+1}, it follows that $K_{n+1} \subseteq A_{n+1} \cup B_{n+1}$, as desired. Therefore A_{n+1} and B_{n+1} must satisfy property (b).

Consider property (c) next. Let $x, y \in A_{n+1}$ and suppose $x \neq y$. We need to show that $d(x, y) \geq 2$. By the construction of A_{n+1}, $x =< a, 0 >$ or $x =< b, 1 >$, and $y =< c, 0 >$ or $y =< e, 1 >$, where $a, c \in A_n$ and $b, e \in B_n$. Therefore there are four cases for x and y.

Suppose $x =< a, 0 >$ and $y =< c, 0 >$ with $a, c \in A_n$. Since $x \neq y$ it must be the case that $a \neq c$. Therefore by property (c) for A_n, $d(a, c) \geq 2$. By the definition of distance, it follows that $d(< a, 0 >, < c, 0 >) = d(a, c)$ and therefore $d(x, y) \geq 2$. A similar argument applies to $x =< b, 1 >$ and $y =< e, 1 >$, with $b, e \in B_n$, showing $d(x, y) \geq 2$ holds in this case too.

Suppose $x =< a, 0 >$ and $y =< e, 1 >$ with $a \in A_n$ and $e \in B_n$. By the definition of distance, it follows that $d(< a, 0 >, < e, 1 >) = 1 + d(a, e)$. But, since A_n and B_n satisfy property (d), and $a \in A_n$, $e \in B_n$, we have $d(a, e) \geq 1$. Therefore $d(x, y) \geq 2$ in this case. A similar argument applies to $x =< b, 1 >$ and $y =< c, 0 >$, with $b \in B_n$ and $c \in A_n$, showing $d(x, y) \geq 2$ holds in this case too.

Since these are all the possible cases for $x, y \in A_{n+1}$, it follows that $d(x, y) \geq 2$ for all distinct elements x and y in A_{n+1}, as required. A similar argument applies to B_{n+1}. Therefore property (c) must hold for A_{n+1} and B_{n+1}.

Finally, we are left with property (d). Let $x \in A_{n+1}$. Then by the construction of A_{n+1} there are two cases, namely $x =< a, 0 >$ or $x =< b, 1 >$ for some element $a \in A_n$ or some $b \in B_n$. Suppose, $x =< a, 0 >$ with $a \in A_n$. Let $y =< a, 1 >$. Then by construction $y \in B_{n+1}$. Moreover $d(x, y) = d(< a, 0 >, < a, 1 >) = 1 + d(a, a) = 1 + 0 = 1$. A similar argument applies to the case $x =< b, 1 >$ with $b \in B_n$, showing that $y =< b, 0 > \in B_{n+1}$ satisfies $d(x, y) = 1$. Therefore we have shown that for any $x \in A_{n+1}$ there exists a $y \in B_{n+1}$ such that $d(x, y) = 1$. A similar argument shows that the roles of A_{n+1} and B_{n+1} can also be reversed. Therefore we have proven that A_{n+1} and B_{n+1} satisfy property (d).

Therefore A_{n+1} and B_{n+1} satisfy all properties (a-d), and hence $S(n + 1)$ is true.

It follows by mathematical induction that $S(n)$ is true for all integers $n \geq 1$.

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6. Consider the set of binary strings of length n, namely K_n, along with the distance function $d(x, y)$, as defined in problem 5. Define the parity function $p : K_n \rightarrow \{even,\ odd\}$ to have the value $p(x) = even$ when the binary string x has an even number of 1's, and $p(x) = odd$ otherwise. We say x has even (or odd) parity if and only if $p(x) = even$ (or $p(x) = odd$, respectively).

   Let $n \geq 1$ be a natural number. Suppose $A \subseteq K_n$ such that, for any $x, y \in A$, either $x = y$ or $d(x, y) \geq 2$. Prove that both of the following statements are true:

   (a) $|A| \leq 2^{n-1}$,

   (b) If $|A| = 2^{n-1}$ then the elements in A all have the same parity, that is, they are all even parity or all odd parity.

   For a change, you don't need to use mathematical induction for this question.

The proof of both parts will rely on the properties of a graph formed using elements of K_n as the nodes. Two nodes in the graph, say corresponding to elements $x, y \in K_n$, are connected by an edge if and only if x and y differ in exactly one place, that is, $d(x, y) = 1$. An important property of this graph is given in Lemma 6.1 below.

Lemma 6.1. Every node in this graph defined on K_n is an endpoint of exactly n edges.

Proof of Lemma 6.1. Consider any node in the graph, that is, any $x =< x_1, x_2, \ldots, x_n >\in K_n$. Define the set $N(x)$ to be

$$N(x) = \{y : y =< x_1, \ldots, x_{i-1}, \bar{x}_i, x_{i+1}, \ldots, x_n >, \\ \text{where } \ 1 \leq i \leq n, \quad \text{and } \ \bar{x}_i \neq x_i\}.$$

Here $\bar{x}_i$ denotes the complement of the bit x_1, that is if $x_i = 0$ then $\bar{x}_i = 1$ and vice versa. For each $y \in N(x)$ it follows from the definition of distance that $d(x, y) = 1$. So there is an edge in the graph between x and y.

Moreover, by the definition of distance, if $y =< y_1, y_2, \ldots, y_n >\in K_n$ is such that $d(x, y) = 1$ then there must be exactly one index i at which $x_i \neq y_i$. Since $x_i, y_i \in \{0, 1\}$ it follows that $y_i = \bar{x}_i$. Therefore $y \in N(x)$. Therefore $N(x)$ is the set of *all* elements in K_n that are a distance 1 away from x. These elements are all the nodes connected to x by an edge in the graph. We refer to $N(x)$ as the set of neighbours of x.

The number of edges terminating at x is therefore $|N(x)|$. And from the construction of $N(x)$ we have $|N(x)| = n$. Since this is true for an arbitrary $x \in K_n$, the lemma follows.

Proof of part a. We will prove this by contradiction. Suppose $A \subseteq K_n$ with the properties that $|A| > 2^{n-1}$ and, for every two distinct elements $x, y \in A$, $d(x, y) \geq 2$. Let $B = K_n - A$ be the complementary set to A. Then $|B| = |K_n| - |A| < 2^n - 2^{n-1} = 2^{n-1}$. (Here we have used $|K_n| = 2^n$, which follows from problem 5.)

Let $x \in A$ and consider the set of neighbours $N(x)$. By definition of $N(x)$, if $y \in N(x)$ then $d(x, y) = 1$ and $x \neq y$. Therefore y cannot be an element of A. Thus $N(x) \subseteq B$ for each $x \in A$.

Let m equal the number of edges in the graph on K_n between elements in A and elements in B. We have shown above that each element $x \in A$ has exactly n distinct neighbours in B. Thus $m = |A|n > n2^{n-1}$.

But, by Lemma 6.1, each element of B has exactly n neighbours and thus m, the total number of edges between A and B, must be bounded by $m \leq |B|n$. From above we know $|B| < 2^{n-1}$. Therefore we have $m < n2^{n-1}$, contradicting the inequality $m > n2^{n-1}$ derived previously.

Therefore $|A| \leq 2^{n-1}$, proving part a.

For part b we will use the following lemma.

Lemma 6.2. Let $x \in K_n$ and $y \in N(x)$. Then the parities $p(x)$ and $p(y)$ are different (i.e. one is even and the other is odd).

Proof of Lemma 6.2. By definition of $N(x)$, y must be the same sequence as x but with one bit changed. That is, for some $i \in \{1, 2, \dots, n\}$, $x =< x_1, \dots, x_n >$ and $y =< y_1, \dots, y_n >$ with $y_k = x_k$ for $k \neq i$ and $y_i = \bar{x}_i$. If $x_i = 0$ then $\bar{x}_i = 1$ and y has exactly one more bit equal to one than x does. Therefore the parity of x and y must be different. Similarly, if $x_i = 1$ then $y_i = \bar{x}_i = 0$ and y has exactly one fewer bits equal to one than x does. Again the parity must be different. Since 0 and 1 are the only two possible values for x_i, the result follows.

Proof of part b. Suppose $A \subseteq K_n$ with the properties that $|A| = 2^{n-1}$ and, for every two distinct elements $x, y \in A$, $d(x, y) \geq 2$. Define $B = K_n - A$ to be the complementary set to A. Moreover, define

$$\begin{aligned} A_e =& \{x : x \in A, \text{ and } p(x) = even\}, \\ A_o =& \{x : x \in A, \text{ and } p(x) = odd\}, \\ B_e =& \{x : x \in B, \text{ and } p(x) = even\}, \\ B_o =& \{x : x \in B, \text{ and } p(x) = odd\}. \end{aligned}$$

By the definitions of A, B and parity, it follows that these four sets are all disjoint and their union is K_n. In particular, since $|K_n| = 2^n$ (see problem 5), $|B| = 2^n - 2^{n-1} = 2^{n-1}$. Therefore we have

$$|A| = |A_e| + |A_o| = 2^{n-1}, \tag{6}$$
$$|B| = |B_e| + |B_o| = 2^{n-1}. \tag{7}$$

Let $x \in A_e$ and consider the set of neighbours $N(x)$. By the definition of A, any other element in A must have a distance of at least 2 from x, and therefore $N(x) \cap A = \emptyset$. Thus $N(x) \subseteq B$. Moreover, by Lemman 6.2, since $x \in A_e$ has even parity, $y \in N(x)$ must have odd parity.

For any set $S \subseteq K_n$ we define

$$N(S) = \{y : y \in K_n \text{ such that there exists an } x \in S \; with \; d(x, y) = 1\}.$$

Thus we have shown above that $N(A_e) \subseteq B_o$. Similarly, we can show that $N(A_o) \subseteq B_e$.

Since each element of A_e has n neighbours in B_o, there are $|A_e|n$ edges between A_e and B_o. Since each element in K_n has at most n neighbours (by Lemma 6.1), it follows that there are no more than $|B_o|n$ edges between A_e and B_o. Therefore $|B_o| \geq |A_e|$. A similar argument shows that $|B_e| \geq |A_o|$. However, from equations (6) and (7), $|A_e| + |A_o| = |B_e| + |B_o|$. Together with the previous inequalities we find that it must be the case that

$$|B_o| = |A_e| \text{ and } |B_e| = |A_o|.$$

Finally, since there are $|B_o|n$ edges with endpoints in B_o, and we know that there are $|A_e|n$ edges between A_e and B_o with $|A_e| = |B_o|$, it follows that *all* the edges with endpoints in B_o must be between elements in A_e and B_o. That is, there can be no edge between B_o and B_e. Therefore $N(B_o) \subseteq A_e$. Similarly, we can show that $N(B_e) \subseteq A_o$. Together with the relations $N(A_e) \subseteq B_o$ and $N(A_o) \subseteq B_e$ proved above, we find that

$$N(A_e) = B_o,\ N(B_o) = A_e,\ N(A_o) = B_e,\ N(B_e) = A_o. \tag{8}$$

We need to prove that either A_e or A_o is empty. We will do this by contradiction.

Suppose A_e and A_o are both non-empty. Let $x =< x_1, \ldots, x_n >\in A_e$ and $y =< y_1, \ldots, y_n >\in A_o$. Consider the sequence $< e^0, e^1, \ldots, e^n >$ with $e^0 = x$, $e^n = y$, and $e^k =< y_1, \ldots, y_k, x_{k+1}, \ldots, x_n >$ for $1 \leq k < n$. Since $K_n = A_e \cup A_o \cup B_e \cup B_o$ and these four subsets are all disjoint, we must have e^k in precisely one of these four subsets of K_n for each k. Also, for any $k \in \{1, \ldots n\}$, either $y_k \neq x_k$, in which case $e^{k-1} \neq e^k$ and $d(e^{k-1}, e^k) = 1$, or $y_k = x_k$ and $e^{k-1} = e^k$. Therefore $e^k \in N(e^{k-1}) \cup \{e^{k-1}\}$ for all $k \in \{1, \ldots n\}$.

Define $L = \{k : e^k \in A_o \cup B_e\}$. Notice that $e^0 = x \in A_e$ implies $0 \notin L$, and $e^n = y \in A_o$ implies $n \in L$. Therefore L is a non-empty subset of the natural numbers. Thus it must have a minimal element $j \in L$. Since $0 \notin L$ the minimal element cannot be 0, so $j > 0$. By the definition of L we then have $e^{j-1} \in A_e \cup B_o$ and $e^j \in A_o \cup B_e$. In particular $e^{j-1} \neq e^j$ and therefore $e^j \in N(e^{j-1})$.

Therefore we have shown $e^{j-1} \in A_e \cup B_o$, equation (8) holds, and $e^j \in N(e^{j-1})$. Together these imply $e^j \in N(A_e) \cup N(B_o) = A_e \cup B_o$. But, since the sets A_e, A_o, B_e and B_o are all disjoint, this contradicts $e^j \in A_o \cup B_e$.

Therefore one of A_e or A_o must be empty, completing the proof.