Assignment 1: Mathematical Induction.

Due: 10 a.m., Thurs., Jan. 30

This assignment is worth 10 percent of the total marks for this course.

In this assignment you will practice proving various statements, mostly using mathematical induction. We will mark only a (secret) subset of the questions below. As always in this course, your proofs will be marked for correctness, along with brevity and clarity.

Your answers can be handwritten. Use standard 8.5 by 11 inch paper. Please staple all the sheets together, and hand them to your tutor at the **beginning** of the tutorial on the due date. If you cannot make it to that tutorial, then leave your assignment at your instructor's office (Pratt, Room 283) before 10am on the due date. Since the tutors will be discussing the solutions in the tutorial immediately after your assigments are due, we will not accept late assignments (the course homepage describes what you should do in case of medical or other emergencies which prevent you from completing an assignment on time).

1. Consider the function $f(n)$ defined by

$$\begin{aligned} f(0) &= 5, \\ f(n) &= f(n-1) + 6n - 3, \quad \text{for } n \geq 1. \end{aligned} \tag{1}$$

Using mathematical induction, prove that $f(n) < 10n^2$ for all natural numbers $n \geq 1$. (Since we want you to practice induction, proofs which do not rely on induction will get zero marks.)

2. Consider the function $f(n)$ defined by

$$\begin{aligned} f(0) &= 5, \\ f(1) &= 4, \\ f(n) &= 3f(\lfloor n/2 \rfloor) + 2^n, \quad \text{for } n \geq 2. \end{aligned} \tag{2}$$

Here $\lfloor n/2 \rfloor$ equals the largest natural number less than or equal to $n/2$. So $\lfloor 3/2 \rfloor = 1$, $\lfloor 4/2 \rfloor = 2$, and so on. Using mathematical induction, prove that $f(n) \leq 2^{n+2}$ for all $n \geq 1$. (Since we want you to practice induction, proofs which do not rely on induction will get a mark of zero.)

3. Consider the function $f(n)$ defined by

$$\begin{aligned} f(0) &= 0, \\ f(1) &= 1, \\ f(n) &= f(n-1) + f(n-2), \quad \text{for } n \geq 2. \end{aligned} \tag{3}$$

Using mathematical induction, prove that

$$f(n)f(n+1) = \sum_{k=0}^{n} f(k)^2, \tag{4}$$

for all natural numbers $n \geq 0$. (Since we want you to practice induction, proofs which do not rely on induction will receive zero marks.)

4. The height of a non-empty tree is defined to be the maximum number of edges in any path from the root of the tree to a leaf node. For an empty tree, we define the height to be -1. Prove that, for each integer $h \geq -1$, if n is the number of nodes in a full binary tree of height h then $n \leq 2^{h+1} - 1$. Your proof must rely on mathematical induction. (Note: The course notes defines a full binary tree as a binary tree for which every node has either 0 or 2 children. This does not imply, for example, that the left and right subtrees of the root node of a full binary tree have the same height.)

5. Let K_n denote the set of all binary strings of length n. That is,

$$K_n = \{a \ : \ a = < a_1, a_2, \ldots, a_n >, \text{ with } a_i = 0 \text{ or } 1 \text{ for each } i = 1 \ldots n\}.$$

Here $< a_1, a_2, \ldots, a_n >$ denotes a sequence, as defined in Chapter 0 of the course notes.

Suppose a and b are elements of K_n. Let $d(a, b)$ be the distance between a and b, which is defined to be the number of indices i at which $a_i \neq b_i$. In particular,

$$d(a, b) = |\{i \ : \ 1 \leq i \leq n \text{ and } a_i \neq b_i, \text{where } a = < a_1, a_2, \ldots, a_n >, b = < b_1, b_2, \ldots, b_n >\}|. \tag{5}$$

For each natural number $n \geq 1$, prove that there exist two sets A_n and B_n for which all of the following properties are satisfied:

(a) $|A_n| = |B_n| = 2^{n-1}$.

(b) $K_n = A_n \cup B_n$ and $\phi = A_n \cap B_n$.

(c) Any two distinct elements of A_n are at least a distance of 2 apart, and similarly for any two distinct elements of B_n. That is, if $x, y \in A_n$ with $x \neq y$ then $d(x, y) \geq 2$. Similarly, if $x, y \in B_n$ with $x \neq y$ then $d(x, y) \geq 2$.

(d) For each element $x \in A_n$ there exists $y \in B_n$ such that $d(x, y) = 1$. (Note the choice of y may depend on the x.) Similarly, for each element $x \in B_n$ there exists $y \in A_n$ such that $d(x, y) = 1$.

Your proof must rely on mathematical induction.

6. Consider the set of binary strings of length n, namely K_n, along with the distance function $d(x, y)$, as defined in problem 5. Define the parity function $p : K_n \to \{even,\ odd\}$ to have the value $p(x) = even$ when the binary string x has an even number of 1's, and $p(x) = odd$ otherwise. We say x has even (or odd) parity if and only if $p(x) = even$ (or $p(x) = odd$, respectively).

   Let $n \geq 1$ be a natural number. Suppose $A \subseteq K_n$ such that, for any $x, y \in A$, either $x = y$ or $d(x, y) \geq 2$. Prove that both of the following statements are true:

   (a) $|A| \leq 2^{n-1}$,

   (b) If $|A| = 2^{n-1}$ then the elements in A all have the same parity, that is, they are all even parity or all odd parity.

   For a change, you don't need to use mathematical induction for this question.