Utilizing Optical Aberrations for Extended-Depth-of-Field Panoramas – Supplementary Material

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1 Deriving the Lens Model

1.1 Deriving Eq. (2)

Assume the cone of rays exiting an idealized point light source forms a focused image at $(x_f, 0, f)$. In this section, we derive the paraxial intersection of one such ray $(s, t, 0)$ with the front lens, knowing that it intersects the back lens at $(u, v, 0)$, as shown in Fig. 11.

After refracted by the front lens, the cone of light exiting the point light source, are focused at a virtual "object point". A thin lens does not bend a ray passing through its center, the ray path from this point to $(0, 0, 0)$ to $(x_f, 0, f)$ is straight. Therefore we denote this virtual "object point" as $(m_B x_f, 0, m_B f)$ by similar triangles.

Now this gives us an expression of $(s, t, -w)$, based on the fact that it is aligned on the same line with $(u, v, 0)$ and $(m_B x_f, 0, m_B f)$.

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + \frac{-w}{m_B f}\left[\begin{pmatrix} m_B x_f \\ 0 \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix}\right] = \left[1 + \frac{w}{m_B f}\right]\begin{pmatrix} u \\ v \end{pmatrix} + \frac{-w}{f}\begin{pmatrix} x_f \\ 0 \end{pmatrix} \tag{14}$$

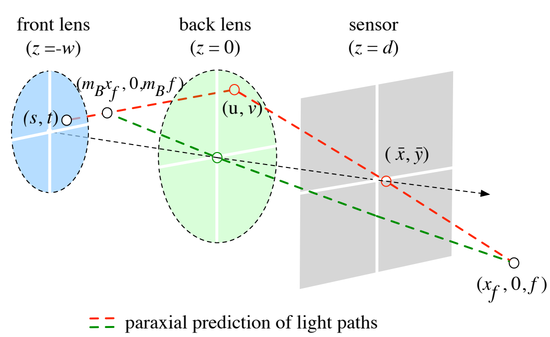

Fig. 11: Derivation of Eq. (2).

Denote

$$m_P \doteq 1 + \frac{w}{m_B f} \tag{15}$$

$$s_0 \doteq \frac{-w}{f} x_f \tag{16}$$

Eq. (14) becomes

$$\begin{pmatrix} s \\ t \end{pmatrix} = m_P \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} s_0 \\ 0 \end{pmatrix} \tag{17}$$

Because the virtual object $(m_B x_f, 0, m_B f)$ is imaged by the back lens at $(x_f, 0, f)$, by the thin lens law

$$\frac{1}{f} + \frac{1}{-m_B f} = \frac{1}{F} \tag{18}$$

Substituting Eq. (18) into Eq. (15), we have

$$m_P \doteq 1 + \frac{w}{m_B f} = w(1/w + \frac{1}{m_B f}) = w(1/w + 1/f - 1/F) \tag{19}$$

Thus, Eqs. (17), (19) and (16) give Eq. (2) in the paper:

$$\begin{pmatrix} s \\ t \end{pmatrix} = m_P \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} s_0 \\ 0 \end{pmatrix}, \text{ where } m_P = w\left(\frac{1}{w} + \frac{1}{f} - \frac{1}{F}\right) \text{ and } s_0 = \frac{-x_f w}{f}$$

1.2 Deriving Eq. (4) and Eq. (5)

In this section we derive the expression for the ray displacement on the image plane due to Seidel aberrations as a function of back lens intersection (u, v) and the expressions for the Seidel coefficients as a function of focal distance f and lens parameters. As shown in Fig. 12, this is by decomposing the ray displacement into two parts: the displacement due to the aberration of the front lens $(\dot{x}, \dot{y}) - (\bar{x}, \bar{y})$, and the displacement due to the aberration of the back lens $(x', y') - (\dot{x}, \dot{y})$.

First we locate $(\bar{x}, \bar{y})$, the sensor intersection of the refracted ray under paraxial approximation. Since $(u, v, 0)$, $(\bar{x}, \bar{y}, d)$ and $(x_f, 0, f)$ are aligned on the same line

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \left(1 - \frac{d}{f}\right) \begin{pmatrix} u \\ v \end{pmatrix} + \frac{d}{f} \begin{pmatrix} x_f \\ 0 \end{pmatrix} \tag{20}$$

Consider an virtual point light source at $(s, t, -w)$. Among rays exiting this light source, the ray passing through $(0, 0, 0)$ is not bent, therefore we can assume the image of this point light source to be at $(m'_B s, m'_B t, -m'_B w)$.

By the thin lens law

$$\frac{1}{w} + \frac{1}{-m'_B w} = \frac{1}{F} \tag{21}$$

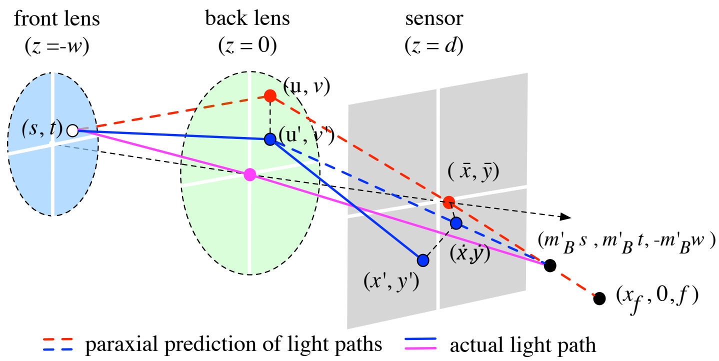

Fig. 12: Derivation of Eq. (4) and Eq. (5)

Now consider another paraxial ray path $(s, t, -w) \to (u', v', 0) \to (m'_B s, m'_B t, -m'_B w)$. Assume this ray intersects the back lens at $(\dot{x}, \dot{y}, d)$. By similar triangles

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = (1 - \frac{d}{-m'_B w}) \left[\begin{pmatrix} u' \\ v' \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} \right] = \left(1 + d \frac{F-w}{Fw} \right) \left[\begin{pmatrix} u' \\ v' \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} \right] \tag{22}$$

Because the angular aberrations are cubic polynomial of ray height,

$$\begin{pmatrix} u' \\ v' \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} = w \cdot c_F (s^2 + t^2) \begin{pmatrix} s \\ t \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = d \cdot c_B (u'^2 + v'^2) \begin{pmatrix} u' \\ v' \end{pmatrix} \approx d \cdot c_B (u^2 + v^2) \begin{pmatrix} u \\ v \end{pmatrix} \tag{24}$$

Therefore, the overall lateral ray displacement on the image plane is

$$\begin{aligned} \begin{pmatrix} \Delta x(u,v) \\ \Delta y(u,v) \end{pmatrix} \doteq \begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} &= \left[\begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \right] + \left[\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \right] \\ &= d \cdot c_B (u^2 + v^2) \begin{pmatrix} u \\ v \end{pmatrix} + (w + d \frac{F-w}{F}) \cdot c_F (s^2 + t^2) \begin{pmatrix} s \\ t \end{pmatrix} \end{aligned} \tag{25}$$

From Eq. (17),

$$\begin{aligned} (s^2 + t^2) \begin{pmatrix} s \\ t \end{pmatrix} &= \left((m_P u + s_0)^2 + (m_P v)^2 \right) \begin{pmatrix} m_P u + s_0 \\ m_P v \end{pmatrix} \\ &= m_P^3 (u^2 + v^2) \begin{pmatrix} u \\ v \end{pmatrix} + m_P^2 s_0 \begin{pmatrix} 3u^2 + v^2 \\ 2uv \end{pmatrix} + m_P s_0^2 \begin{pmatrix} 3u \\ v \end{pmatrix} + s_0^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned} \tag{26}$$

Substituting Eq. (26) into Eq. (25), we arrive at Eq. (4), the expression for the lateral ray displacement on the image plane:

$$\begin{pmatrix} \Delta x(u,v) \\ \Delta y(u,v) \end{pmatrix} \doteq \begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \underbrace{\alpha_1 (u^2+v^2) \begin{pmatrix} u \\ v \end{pmatrix}}_{\text{spherical}} + \underbrace{\alpha_2 x_f \begin{pmatrix} 3u^2+v^2 \\ 2uv \end{pmatrix}}_{\text{coma}}$$
$$+ \underbrace{\alpha_3 x_f^2 \begin{pmatrix} u \\ 0 \end{pmatrix}}_{\text{astigmatism}} + \underbrace{\alpha_4 x_f^2 \begin{pmatrix} u \\ v \end{pmatrix}}_{\text{field curvature}} + \underbrace{\alpha_5 x_f^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{field distortion}}$$

and Eq. (5), the expression of Seidel coefficients as a function of lens parameters and focus setting f:

$$\alpha_1 = dc_B + c_F(w + d\frac{F-w}{F})m_P^3, \quad \alpha_2 = -c_F(w + d\frac{F-w}{F})m_P^2(w/f)$$
$$\alpha_3 = 2c_F(w + d\frac{F-w}{F})m_P(w/f)^2, \quad \alpha_4 = c_F(w + d\frac{F-w}{F})m_P(w/f)^2$$
$$\alpha_5 = -c_F(w + d\frac{F-w}{F})(w/f)^3$$

2 Deriving Eq. (14)

By the convolution theorem, the formation of the blurry path of depth λ can be written in the Fourier domain:

$$\mathcal{F}[\varphi_j](\mu,\nu) = \mathcal{F}[k_\lambda^j](\mu,\nu)\mathcal{F}[\psi](\mu,\nu) + \mathcal{F}[n](\mu,\nu), \forall \mu,\nu,j \tag{27}$$

In the following we denote Φ_j and Ψ as the 1D vector forms of $\mathcal{F}[\varphi_j]$ and $\mathcal{F}[\psi]$, and K_λ^j as a diagonal matrix with the non-zero entires being pixels in $\mathcal{F}[k_\lambda^j]$.

Since n is random Gaussian noise of variance η^2, the probability of Φ_j conditioned on Ψ and λ is

$$\Pr(\Phi_j;\Psi,\lambda) \sim \mathcal{N}\left(\Phi_j | K_d^j \Psi, \eta^2 I_P\right) \tag{28}$$

with I_D being the $D \times D$ unit matrix. Here P is the number of pixels in ψ.

Assuming that Ψ is a Gaussian of zero mean and isotropic variance S,

$$\Pr(\Psi) \sim \mathcal{N}(0, SI_P) \tag{29}$$

From Eq. (28) and (29), the frequencies in the observed images

$$\Phi = \left[\Phi_1, \cdots, \Phi_j, \cdots, \Phi_N\right]$$

are also Gaussian:

$$\Pr(\Phi;\lambda) = \mathcal{N}(0, \eta^2 I_{NP} + SK_\lambda^T \overline{K_\lambda}) \tag{30}$$

where K_λ is a matrix of $P \times NP$ elements corresponding to the horizontal concatenate of vectorized OTFs under the depth hypothesis λ

$$K_\lambda = \left[K_\lambda^1, \cdots, K_\lambda^j, \cdots, K_\lambda^N\right] \tag{31}$$

In the following we define K_1 and K_2 as the K_λ for depth hypothesis λ_1 and λ_2. Now consider the distribution of Φ under these two depth hypothesis, their variance are

$$\Sigma_1 = \eta^2 I_{NP} + S K_1^T \overline{K_1}, \quad \Sigma_2 = \eta^2 I_{NP} + S K_2^T \overline{K_2} \tag{32}$$

The Kullback-Leibler(KL) divergence between these two Gaussian distributions are therefore

$$\begin{aligned} \mathrm{KL}(\lambda_1, \lambda_2) &= \int \Pr(\Phi; \lambda_1) \Big(\log \Pr(\Phi; \lambda_1) - \log \Pr(\Phi; \lambda_2) \Big) \lambda \Phi \\ &= \frac{1}{2} \left(\log |\Sigma_1^{-1} \Sigma_2| + \mathrm{tr}(\Sigma_1^{-1} \Sigma_2) - NP \right) \end{aligned} \tag{33}$$

Now, we compute the determinant and trace of $\Sigma_1^{-1}\Sigma_2$. By the matrix determinant formula

$$|\Sigma_1| = \eta^{2NP} \cdot |I_P + \eta^{-2} S \overline{K_1} K_1^T| \tag{34}$$

$$|\Sigma_2| = \eta^{2NP} \cdot |I_P + \eta^{-2} S \overline{K_2} K_2^T| \tag{35}$$

From Eq. (31),

$$\overline{K_\lambda} K_\lambda^T = \mathrm{diag}(\sum_j |\mathcal{F}[k_\lambda^j]|^2) \tag{36}$$

therefore

$$|\Sigma_1| = \prod_{\mu,\nu} (\eta^2 + \sum_j |\mathcal{F}[k_1^j](\mu, \nu)|^2) \tag{37}$$

$$|\Sigma_2| = \prod_{\mu,\nu} (\eta^2 + \sum_j |\mathcal{F}[k_2^j](\mu, \nu)|^2) \tag{38}$$

The determinant of $\Sigma_1^{-1}\Sigma_2$ is thus

$$|\Sigma_1^{-1} \Sigma_2| = \frac{|\Sigma_2|}{|\Sigma_1|} = \prod_{\mu,\nu} \left(\frac{\eta^2 + S \sum_j |\mathcal{F}[k_2^j](\mu, \nu)|^2}{\eta^2 + S \sum_j |\mathcal{F}[k_1^j](\mu, \nu)|^2} \right) \tag{39}$$

From the Woodbury matrix identity,

$$\Sigma_1^{-1} = \frac{1}{\eta^2} \left(I_{NP} - \frac{S}{\eta^2} K_1^T (I_P + \frac{S}{\eta^2} \overline{K_1} K_1^T)^{-1} \overline{K_1} \right) \tag{40}$$

Therefore,

$$\begin{aligned}\Sigma_1^{-1}\Sigma_2 &= \left(I_{NP} - \frac{S}{\eta^2}K_1^T(I_P + \frac{S}{\eta^2}\overline{K_1}K_1^T)^{-1}\overline{K_1}\right)\left(I_{NP} + \frac{S}{\eta^2}K_2^T\overline{K_2}\right)\\ &= I_{NP} + \frac{S}{\eta^2}K_2^T\overline{K_2} - \frac{S}{\eta^2}K_1^T(I_P + \frac{S}{\eta^2}\overline{K_1}K_1^T)^{-1}\overline{K_1}\\ &- \left(\frac{S}{\eta^2}\right)^2 K_1^T(I_P + \frac{S}{\eta^2}\overline{K_1}K_1^T)^{-1}\overline{K_1}K_2^T\overline{K_2}\end{aligned} \tag{41}$$

From Eq. (44),

$$\mathrm{tr}(\frac{S}{\eta^2}K_2^T\overline{K_2}) = \frac{S}{\eta^2}\sum_{\mu,\nu}\sum_j |\mathcal{F}[k_2^j](\mu,\nu)|^2 \tag{42}$$

$$\mathrm{tr}(\frac{S}{\eta^2}K_1^T(I_P + \frac{S}{\eta^2}\overline{K_1}K_1^T)^{-1}\overline{K_1}) = \sum_{\mu,\nu}\sum_j \frac{|\mathcal{F}[k_1^j](\mu,\nu)|^2}{\frac{\eta^2}{S} + |\mathcal{F}[k_1^j](\mu,\nu)|^2} \tag{43}$$

Since

$$\overline{K_1}K_2^T = \mathrm{diag}(\sum_j \overline{\mathcal{F}[k_1^j]}\mathcal{F}[k_2^j]) \tag{44}$$

we have

$$\begin{aligned}&\mathrm{tr}\left(\left(\frac{S}{\eta^2}\right)^2 K_1^T(I_{NP} + \frac{S}{\eta^2}\overline{K_1}K_1^T)^{-1}\overline{K_1}K_2^T\overline{K_2}\right)\\ =&\mathrm{tr}\left(\left(\frac{S}{\eta^2}\right)^2 K_1^T\mathrm{diag}\left(\frac{\sum_j \overline{\mathcal{F}[k_1^j]}\mathcal{F}[k_2^j]}{1 + \frac{S}{\eta^2}\sum_j |\mathcal{F}[k_\lambda^j]|^2}\right)\overline{K_2}\right)\\ =&\frac{S}{\eta^2}\sum_{\mu,\nu}\frac{|\sum_j \mathcal{F}[k_1^j](\mu,\nu)\overline{\mathcal{F}[k_2^j](\mu,\nu)}|^2}{\sum_j(\frac{\eta^2}{S} + |\mathcal{F}[k_1^j](\mu,\nu)|^2)}\end{aligned} \tag{45}$$

The trace of $\Sigma_1^{-1}\Sigma_2$ is

$$\begin{aligned}\mathrm{tr}\left(\Sigma_1^{-1}\Sigma_2\right) &= NP + \frac{S}{\eta^2}\sum_{\mu,\nu,j}|\mathcal{F}[k_2^j](\mu,\nu)|^2 - \sum_{\mu,\nu,j}\frac{S|\mathcal{F}[k_1^j]|^2(\mu,\nu)}{\eta^2 + S|\mathcal{F}[k_1^j]|^2(\mu,\nu)}\\ &- \frac{S}{\eta^2}\sum_{\mu,\nu}\frac{|\sum_j |\mathcal{F}[k_1^j](\mu,\nu)\overline{\mathcal{F}[k_2^j](\mu,\nu)}|^2}{\sum_j(\frac{\eta^2}{S} + |\mathcal{F}[k_1^j](\mu,\nu)|^2)}\end{aligned} \tag{46}$$

Substituting Eq. (39) and Eq. (46) into Eq. (33) gives Eq. (14)

$$\begin{aligned}\mathrm{KL}(\lambda_1,\lambda_2) = \frac{1}{2}\sum_{\mu,\nu}\Bigg[\log\left(\frac{\eta^2 + S\sum_j |\mathcal{F}[k_2^j]||^2}{\eta^2 + S\sum_j |\mathcal{F}[k_1^j]||^2}\right) + \frac{S}{\eta^2}\sum_j |\mathcal{F}[k_2^j]|^2\\ - \sum_j \frac{S|\mathcal{F}[k_1^j]|^2}{\eta^2 + S|\mathcal{F}[k_1^j]|^2} - \frac{S}{\eta^2}\frac{S|\sum_j \mathcal{F}[k_1^j]\overline{\mathcal{F}[k_2^j]}|^2}{\sum_j \eta^2 + S|\mathcal{F}[k_1^j]|^2}\Bigg](\mu,\nu)\ .\end{aligned}$$