CSC2515 Fall 2007
Introduction to Machine Learning

Lecture 8 Deep Belief Nets

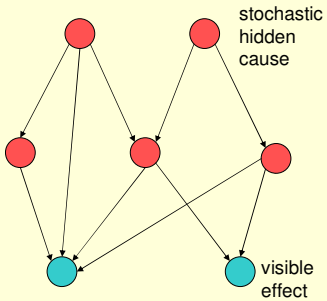
All lecture slides will be available as .ppt, .ps, & .htm at
www.cs.toronto.edu/~hinton

Three ways to combine probability density models

- **Mixture:** A weighted average of the distributions.
  - It can never be sharper than the individual distributions.
- **Product:** Multiply the distributions at each point and then renormalize.
  - Much more powerful than a mixture, but the normalization can make learning difficult.
- **Composition:** Use the values of the latent variables of one model as the data for the next model.
  - Learns multiple layers of representation.
  - We would like to guarantee that the composite model improves every time we add a new layer.

Belief Nets

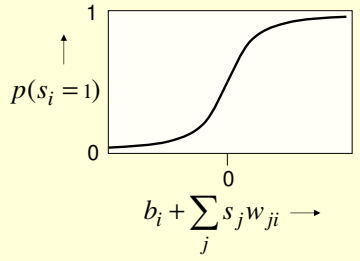
- A belief net is a directed acyclic graph composed of stochastic variables.
- We get to observe some of the variables and we would like to solve two problems:
- The inference problem: Infer the states of the unobserved variables.
- The learning problem: Adjust the interactions between variables to make the network more likely to generate the observed data.



We will use nets composed of layers of stochastic binary variables with weighted connections

Stochastic binary neurons

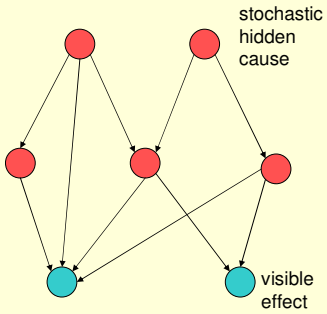
- These have a state of 1 or 0.
- The probability of turning on is determined by the weighted input from other neurons (plus a bias)



$$p(s_i = 1) = \frac{1}{1 + \exp(-b_i - \sum_j s_j w_{ji})}$$

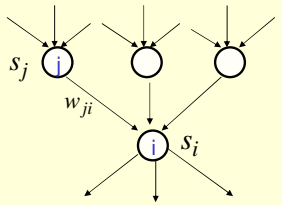
Learning Belief Nets

- It is easy to generate an unbiased example at the leaf nodes, so we can see what kinds of data the network believes in.
- It is hard to infer the posterior distribution over all possible configurations of hidden causes.
- It is hard to even get a sample from the posterior.
- So how can we learn deep belief nets that have millions of parameters?



The learning rule for sigmoid belief nets

- Learning is easy if we can get an unbiased sample from the posterior distribution over hidden states given the observed data.
- For each unit, maximize the log probability that its binary state in the sample from the posterior would be generated by the sampled binary states of its parents.

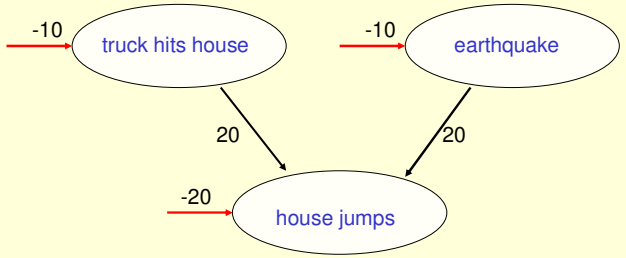


$$p_i \equiv p(s_i = 1) = \frac{1}{1+\exp(-\sum_j s_j w_{ji})}$$

$$\Delta w_{ji} = \varepsilon\, s_j (s_i - p_i)$$

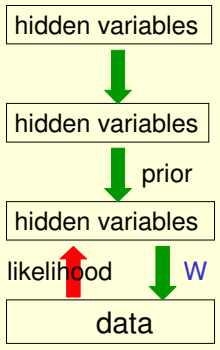
Explaining away (Judea Pearl)

- Even if two hidden causes are independent, they can become dependent when we observe an effect that they can both influence.
  - If we learn that there was an earthquake it reduces the probability that the house jumped because of a truck.



Why it is usually very hard to learn sigmoid belief nets one layer at a time

- To learn W, we need the posterior distribution in the first hidden layer.
- Problem 1: The posterior is typically intractable because of "explaining away".
- Problem 2: The posterior depends on the prior as well as the likelihood.
  - So to learn W, we need to know the weights in higher layers, even if we are only approximating the posterior. All the weights interact.
- Problem 3: We need to integrate over all possible configurations of the higher variables to get the prior for first hidden layer. Yuk!

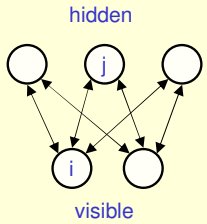


Two types of generative neural network

- If we connect binary stochastic neurons in a directed acyclic graph we get a Sigmoid Belief Net (Radford Neal 1992).
- If we connect binary stochastic neurons using symmetric connections we get a Boltzmann Machine (Hinton & Sejnowski, 1983).
  - If we restrict the connectivity in a special way, it is easy to learn a Boltzmann machine.

Restricted Boltzmann Machines

- We restrict the connectivity to make learning easier.
  - Only one layer of hidden units.
    - We will deal with more layers later
  - No connections between hidden units.
- In an RBM, the hidden units are conditionally independent given the visible states.
  - So we can quickly get an unbiased sample from the posterior distribution when given a data-vector.
  - This is a big advantage over directed belief nets



The Energy of a joint configuration

(ignoring terms to do with biases)

binary state of visible unit i

binary state of hidden unit j

$$E(v,h) = -\sum_{i,j} v_i h_j w_{ij}$$

Energy with configuration v on the visible units and h on the hidden units

weight between units i and j

$$\frac{\partial E(v,h)}{\partial w_{ij}} = -v_i h_j$$

Weights → Energies → Probabilities

- Each possible joint configuration of the visible and hidden units has an energy
  - The energy is determined by the weights and biases (as in a Hopfield net).
- The energy of a joint configuration of the visible and hidden units determines its probability:

$$p(v,h) \propto e^{-E(v,h)}$$

- The probability of a configuration over the visible units is found by summing the probabilities of all the joint configurations that contain it.

Using energies to define probabilities

- The probability of a joint configuration over both visible and hidden units depends on the energy of that joint configuration compared with the energy of all other joint configurations.

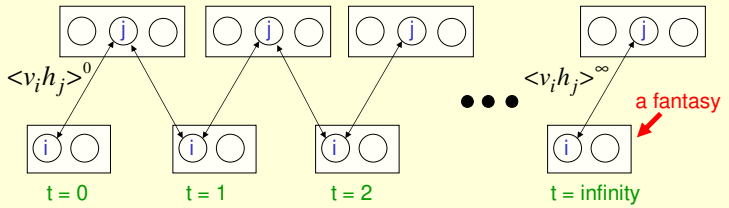
$$p(v,h) = \frac{e^{-E(v,h)}}{\sum_{u,g} e^{-E(u,g)}}$$

partition function

- The probability of a configuration of the visible units is the sum of the probabilities of all the joint configurations that contain it.

$$p(v) = \frac{\sum_{h} e^{-E(v,h)}}{\sum_{u,g} e^{-E(u,g)}}$$

A picture of the maximum likelihood learning algorithm for an RBM

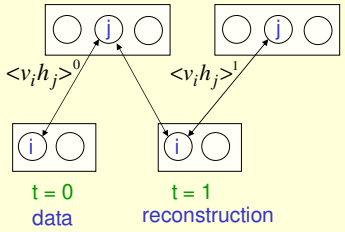


Start with a training vector on the visible units.

Then alternate between updating all the hidden units in parallel and updating all the visible units in parallel.

$$\frac{\partial \log p(v)}{\partial w_{ij}} = <v_i h_j>^0 - <v_i h_j>^\infty$$

A quick way to learn an RBM



Start with a training vector on the visible units.

Update all the hidden units in parallel

Update the all the visible units in parallel to get a "reconstruction".

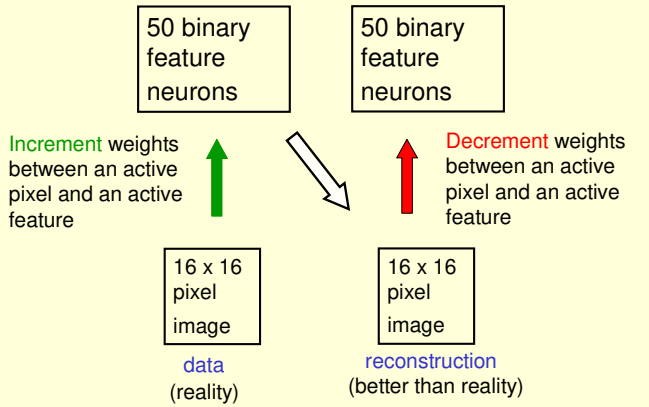
Update the hidden units again.

$$\Delta w_{ij} = \varepsilon \left(<v_i h_j>^0 - <v_i h_j>^1 \right)$$

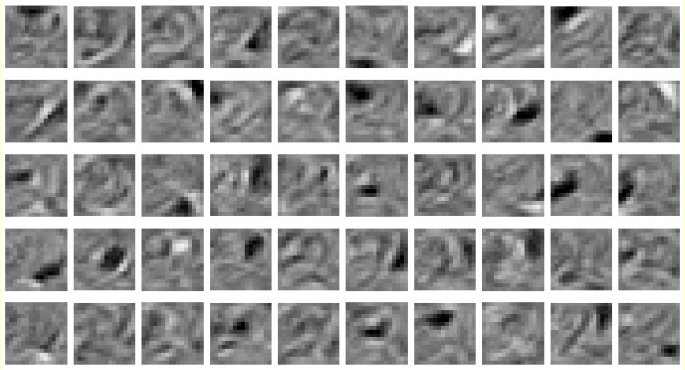
This is not following the gradient of the log likelihood. But it works well.

It is approximately following the gradient of another objective function.

How to learn a set of features that are good for reconstructing images of the digit 2

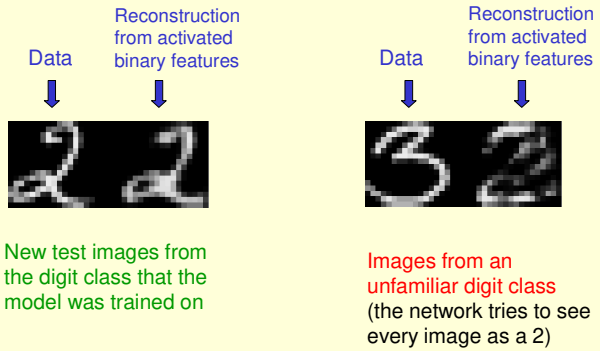


The final 50 x 256 weights



Each neuron grabs a different feature.

How well can we reconstruct the digit images from the binary feature activations?



New test images from the digit class that the model was trained on

Images from an unfamiliar digit class (the network tries to see every image as a 2)

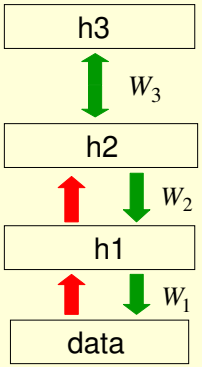
Training a deep network

- First train a layer of features that receive input directly from the pixels.
- Then treat the activations of the trained features as if they were pixels and learn features of features in a second hidden layer.
- It can be proved that each time we add another layer of features we improve a variational lower bound on the log probability of the training data.
  - The proof is slightly complicated.
  - But it is based on a neat equivalence between an RBM and a deep directed model (described later)

The generative model after learning 3 layers

- To generate data:
  1. Get an equilibrium sample from the top-level RBM by performing alternating Gibbs sampling.
  2. Perform a top-down pass to get states for all the other layers.

So the lower level bottom-up connections are not part of the generative model. They are just used for inference.



Why does greedy learning work?

The weights, W, in the bottom level RBM define p(v|h) and they also, indirectly, define p(h).

So we can express the RBM model as

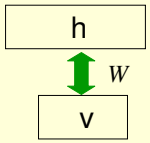
$$p(v) = \sum_h p(h)\ p(v \mid h)$$

If we leave p(v|h) alone and improve p(h), we will improve p(v).

To improve p(h), we need it to be a better model of the aggregated posterior distribution over hidden vectors produced by applying W to the data.

What does each RBM achieve?

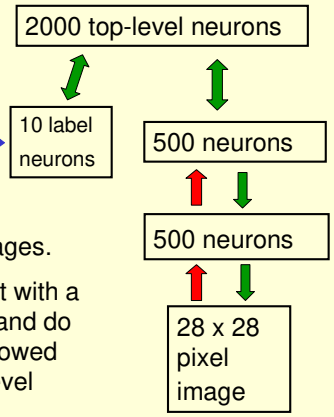
- It divides the task of modeling the data into two tasks and leaves the second task to the next RBM
  - Task 1: Learn generative weights that can convert the posterior distribution over the hidden units into the data.
  - Task 2: Learn to model the posterior distribution over the hidden units that is produced by applying the transpose of the generative weights to the data
    - Task 2 is guaranteed to be easier (for the next RBM) than modeling the original data.



A neural model of digit recognition

The top two layers form an associative memory whose energy landscape models the low dimensional manifolds of the digits.

The energy valleys have names



The model learns to generate combinations of labels and images.

To perform recognition we start with a neutral state of the label units and do an up-pass from the image followed by a few iterations of the top-level associative memory.

Fine-tuning with a contrastive divergence version of the "wake-sleep" algorithm

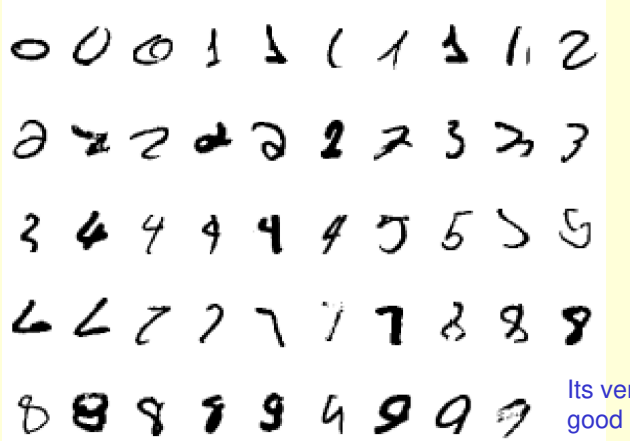
- After learning many layers of features, we can fine-tune the features to improve generation.
- 1. Do a stochastic bottom-up pass
  - Adjust the top-down weights to be good at reconstructing the feature activities in the layer below.
- 2. Do a few iterations of sampling in the top level RBM
  - Use CD learning to improve the RBM
- 3. Do a stochastic top-down pass
  - Adjust the bottom-up weights to be good at reconstructing the feature activities in the layer above.

Show the movie of the network generating digits

(available at www.cs.toronto/~hinton)

Samples generated by letting the associative memory run with one label clamped. There are 1000 iterations of alternating Gibbs sampling between samples.

Examples of correctly recognized handwritten digits that the neural network had never seen before



Its very good

How well does it discriminate on MNIST test set with no extra information about geometric distortions?

- Generative model based on RBM's 1.25%
- Support Vector Machine (Decoste et. al.) 1.4%
- Backprop with 1000 hiddens (Platt) ~1.6%
- Backprop with 500 -->300 hiddens ~1.6%
- K-Nearest Neighbor ~ 3.3%

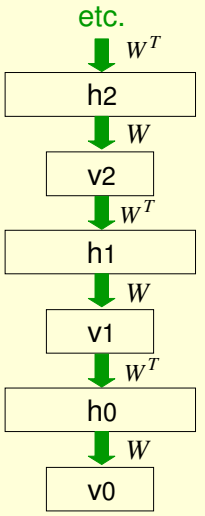
- Its better than backprop and much more neurally plausible because the neurons only need to send one kind of signal, and the teacher can be another sensory input.

Another view of why layer-by-layer learning works

- There is an unexpected equivalence between RBM's and directed networks with many layers that all use the same weights.
  - This equivalence also gives insight into why contrastive divergence learning works.

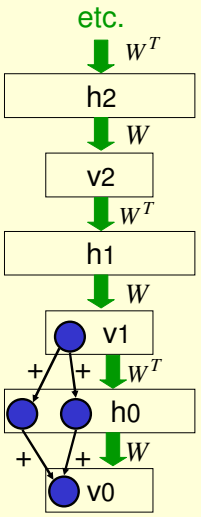
An infinite sigmoid belief net that is equivalent to an RBM

- The distribution generated by this infinite directed net with replicated weights is the equilibrium distribution for a compatible pair of conditional distributions: p(v|h) and p(h|v) that are both defined by W
  - A top-down pass of the directed net is exactly equivalent to letting a Restricted Boltzmann Machine settle to equilibrium.
  - So this infinite directed net defines the same distribution as an RBM.



Inference in a directed net with replicated weights

- The variables in h0 are conditionally independent given v0.
  - Inference is trivial. We just multiply v0 by W transpose.
  - The model above h0 implements a complementary prior.
  - Multiplying v0 by W transpose gives the product of the likelihood term and the prior term.
- Inference in the directed net is exactly equivalent to letting a Restricted Boltzmann Machine settle to equilibrium starting at the data.

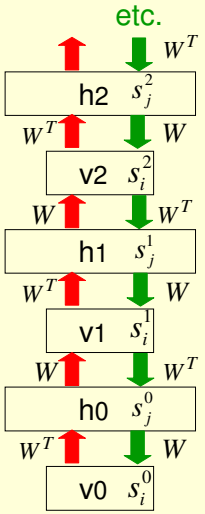


- The learning rule for a sigmoid belief net is:

$$\Delta w_{ij} \propto s_j (s_i - \hat{s}_i)$$

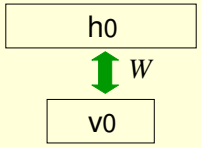
- With replicated weights this becomes:

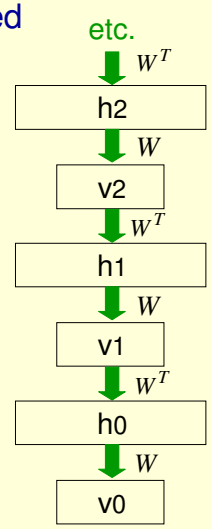
$$s_j^0 (s_i^0 - s_i^1) + s_i^1 (s_j^0 - s_j^1) + s_j^1 (s_i^1 - s_i^2) + \ldots s_j^\infty s_i^\infty$$



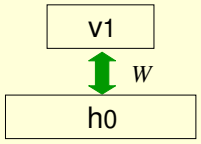
Learning a deep directed network

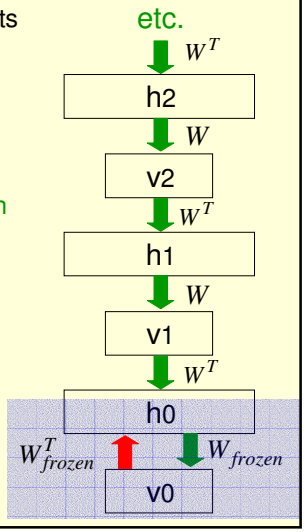
- First learn with all the weights tied
  - This is exactly equivalent to learning an RBM
  - Contrastive divergence learning is equivalent to ignoring the small derivatives contributed by the tied weights between deeper layers.





- Then freeze the first layer of weights in both directions and learn the remaining weights (still tied together).
  - This is equivalent to learning another RBM, using the aggregated posterior distribution of h0 as the data.





What happens when the weights in higher layers become different from the weights in the first layer?

- The higher layers no longer implement a complementary prior.
  - So performing inference using the frozen weights in the first layer is no longer correct.
  - Using this incorrect inference procedure gives a variational lower bound on the log probability of the data.
    - We lose by the slackness of the bound.
- The higher layers learn a prior that is closer to the aggregated posterior distribution of the first hidden layer.
  - This improves the network's model of the data.
    - Hinton, Osindero and Teh (2006) prove that this improvement is always bigger than the loss.

A stack of RBM's

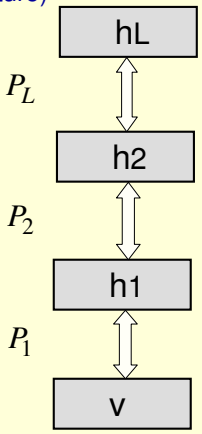
(Yee-Whye Teh's picture)

- Each RBM has the same subscript as its hidden layer.
- Each RBM defines its own distribution over its visible vectors

$$P_l(h_{l-1}) = \frac{\sum_{h_l} \exp(-E(h_{l-1}, h_l))}{Z_l}$$

- Each RBM defines its own distribution over its hidden vectors

$$P_l(h_l) = \frac{\sum_{h_{l-1}} \exp(-E(h_{l-1}, h_l))}{Z_l}$$



The variational bound

Each time we replace the prior over the hidden units by a better prior, we win by the difference in the probability assigned

$$\log p(v) \geq \log P_1(v) + \sum_{l=1}^{l=L-1} \left\langle \log P_{l+1}(h_l) - \log P_l(h_l) \right\rangle_{Q(h_l|v)}$$

Now we cancel out all of the partition functions except the top one and replace log probabilities by goodnesses using the fact that:

$$\log P_l(x) = G_l(x) - \log Z_l$$

$$G(v) = \log \sum_h \exp(-E(v,h))$$

$$G(h) = \log \sum_v \exp(-E(v,h))$$

$$\log p(v) \geq G_1(v) + \sum_{l=1}^{l=L-1} \left\langle G_{l+1}(h_l) - G_l(h_l) \right\rangle_{Q(h_l|v)} - \log Z_L$$

This has simple derivatives that give a more justifiable fine-tuning algorithm than contrastive wake-sleep.

Summary so far

- Restricted Boltzmann Machines provide a simple way to learn a layer of features without any supervision.
  - Maximum likelihood learning is computationally expensive because of the normalization term, but contrastive divergence learning is fast and usually works well.
- Many layers of representation can be learned by treating the hidden states of one RBM as the visible data for training the next RBM (a composition of experts).
- This creates good generative models that can then be fine-tuned.
  - Contrastive wake-sleep can fine-tune generation.

Fine-tuning for discrimination

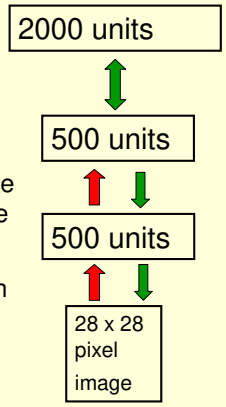
- First learn one layer at a time greedily.
- Then treat this as "pre-training" that finds a good initial set of weights which can be fine-tuned by a local search procedure.
  - Contrastive wake-sleep is one way of fine-tuning the model to be better at generation.
- Backpropagation can be used to fine-tune the model for better discrimination.
  - This overcomes many of the limitations of standard backpropagation.

Why backpropagation works better after greedy pre-training

- Greedily learning one layer at a time scales well to really big networks, especially if we have locality in each layer.
- We do not start backpropagation until we already have sensible weights that already do well at the task.
  - So the initial gradients are sensible and backprop only needs to perform a local search.
- Most of the information in the final weights comes from modeling the distribution of input vectors.
  - The precious information in the labels is only used for the final fine-tuning. It slightly modifies the features. It does not need to discover features.
  - This type of backpropagation works well even if most of the training data is unlabeled. The unlabeled data is still very useful for discovering good features.

First, model the distribution of digit images

The top two layers form a restricted Boltzmann machine whose free energy landscape should model the low dimensional manifolds of the digits.



The network learns a density model for unlabeled digit images. When we generate from the model we get things that look like real digits of all classes.

But do the hidden features really help with digit discrimination?

Add 10 softmaxed units to the top and do backpropagation.

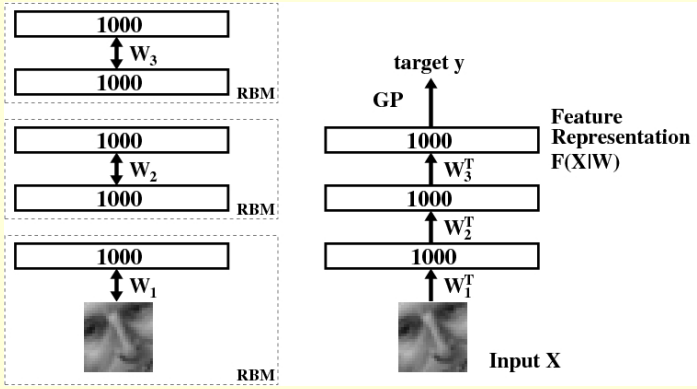
Results on permutation-invariant MNIST task

- Very carefully trained backprop net with one or two hidden layers (Platt; Hinton) 1.6%
- SVM (Decoste & Schoelkopf) 1.4%
- Generative model of joint density of images and labels (+ generative fine-tuning) 1.25%
- Generative model of unlabelled digits followed by gentle backpropagation 1.15%

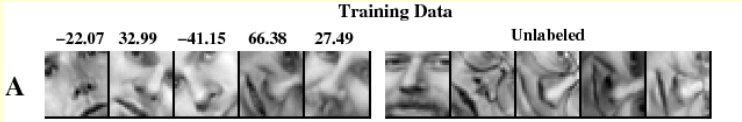
Combining deep belief nets with Gaussian processes

- Deep belief nets can benefit a lot from unlabeled data when labeled data is scarce.
  - They just use the labeled data for fine-tuning.
- Kernel methods, like Gaussian processes, work well on small labeled training sets but are very slow for large training sets.
- So when there is a lot of unlabeled data and only a little labeled data, combine the two approaches:
  - First learn a deep belief net without using the labels.
  - Then apply Gaussian process models to the deepest layer of features. This works better than using the raw data.
  - Use GP's to get the derivatives that are backpropagated through the deep belief net. This is a further win. It allows GP's to fine-tune complicated kernels.

Learning to extract the orientation of a face patch (Ruslan Salakhutdinov)



The training and test sets



100, 500, or 1000 labeled cases 11,000 unlabeled cases

face patches from new people

The root mean squared error in the orientation when combining GP's with deep belief nets

	GP on the pixels	GP on top-level features	GP on top-level features with fine-tuning
100 labels	22.2	17.9	15.2
500 labels	17.2	12.7	7.2
1000 labels	16.3	11.2	6.4

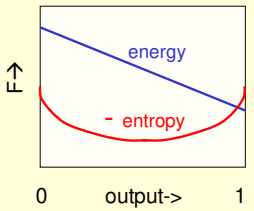
Conclusion: The deep features are much better than the pixels. Fine-tuning helps a lot.

Modeling real-valued data

- For images of digits it is possible to represent intermediate intensities as if they were probabilities by using "mean-field" logistic units.
  - We can treat intermediate values as the probability that the pixel is inked.
- This will not work for real images.
  - In a real image, the intensity of a pixel is almost always almost exactly the average of the neighboring pixels.
  - Mean-field logistic units cannot represent precise intermediate values.

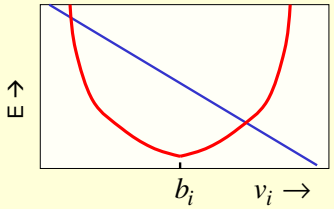
The free-energy of a mean-field logistic unit

- In a mean-field logistic unit, the total input provides a linear energy-gradient and the negative entropy provides a containment function with fixed curvature.
- So it is impossible for the value 0.7 to have much lower free energy than both 0.8 and 0.6. This is no good for modeling real-valued data.



An RBM with real-valued visible units

- Using Gaussian visible units we can get much sharper predictions and alternating Gibbs sampling is still easy, though learning is slower.



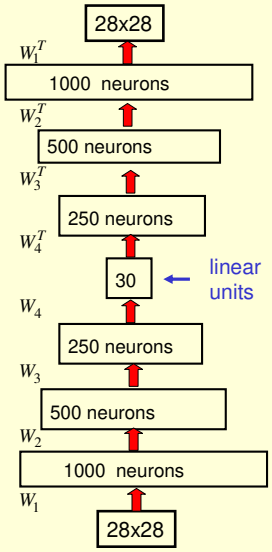
parabolic containment function

energy-gradient produced by the total input to a visible unit

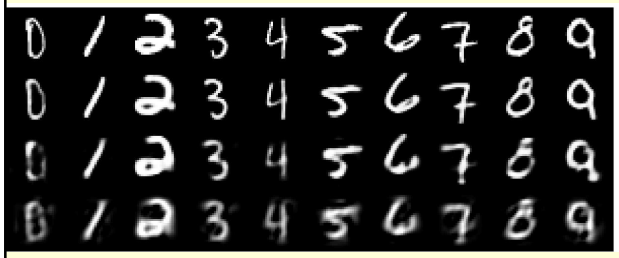
$$E(\mathbf{v},\mathbf{h}) = \sum_{i \,\varepsilon\, vis} \frac{(v_i - b_i)^2}{2\sigma_i^2} - \sum_{j \,\varepsilon\, hid} b_j h_j - \sum_{i,j} \frac{v_i}{\sigma_i} h_j \, w_{ij}$$

Deep Autoencoders
(Ruslan Salakhutdinov)

- They always looked like a really nice way to do non-linear dimensionality reduction:
  - But it is very difficult to optimize deep autoencoders using backpropagation.
- We now have a much better way to optimize them:
  - First train a stack of 4 RBM's
  - Then "unroll" them.
  - Then fine-tune with backprop.



A comparison of methods for compressing digit images to 30 real numbers.



real data

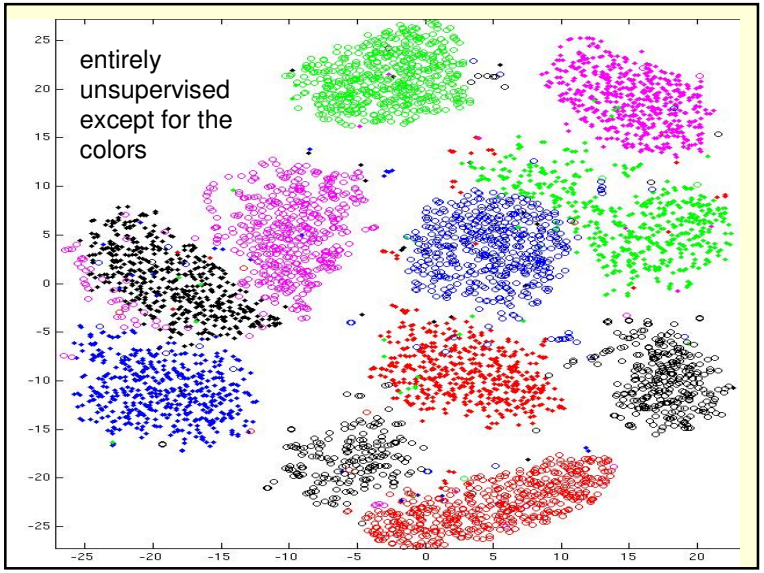
30-D deep auto

30-D logistic PCA

30-D PCA

Do the 30-D codes found by the deep autoencoder preserve the class structure of the data?

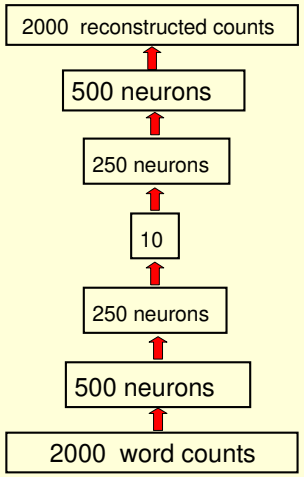
- Take the 30-D activity patterns in the code layer and display them in 2-D using a new form of non-linear multi-dimensional scaling (UNI-SNE)
- Will the learning find the natural classes?



Retrieving documents that are similar to a query document

- We can use an autoencoder to find low-dimensional codes for documents that allow fast and accurate retrieval of similar documents from a large set.
- We start by converting each document into a "bag of words". This a 2000 dimensional vector that contains the counts for each of the 2000 commonest words.

How to compress the count vector

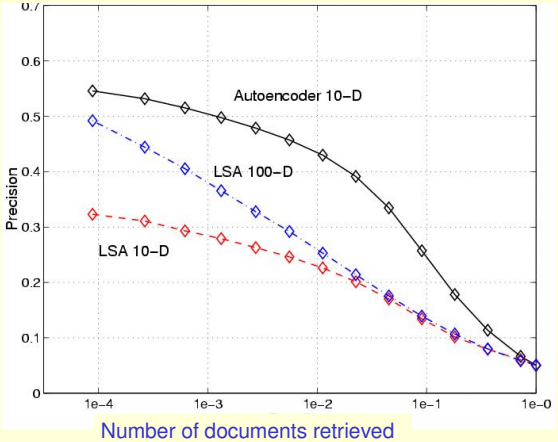


- We train the neural network to reproduce its input vector as its output
- This forces it to compress as much information as possible into the 10 numbers in the central bottleneck.
- These 10 numbers are then a good way to compare documents.

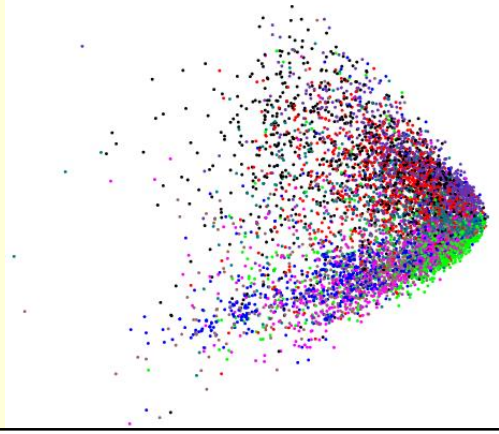
Performance of the autoencoder at document retrieval

- Train on bags of 2000 words for 400,000 training cases of business documents.
  - First train a stack of RBM's. Then fine-tune with backprop.
- Test on a separate 400,000 documents.
  - Pick one test document as a query. Rank order all the other test documents by using the cosine of the angle between codes.
  - Repeat this using each of the 400,000 test documents as the query (requires 0.16 trillion comparisons).
- Plot the number of retrieved documents against the proportion that are in the same hand-labeled class as the query document.

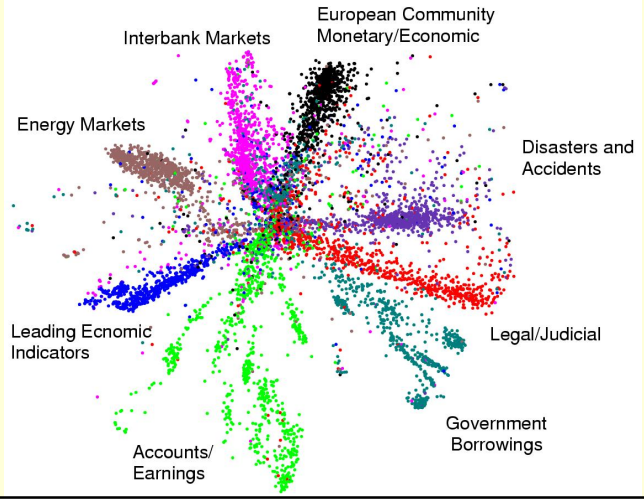
Proportion of retrieved documents in same class as query



First compress all documents to 2 numbers using a type of PCA
Then use different colors for different document categories

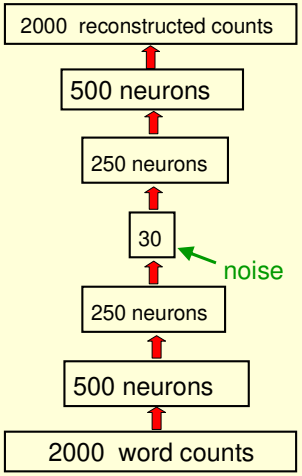


First compress all documents to 2 numbers.
Then use different colors for different document categories

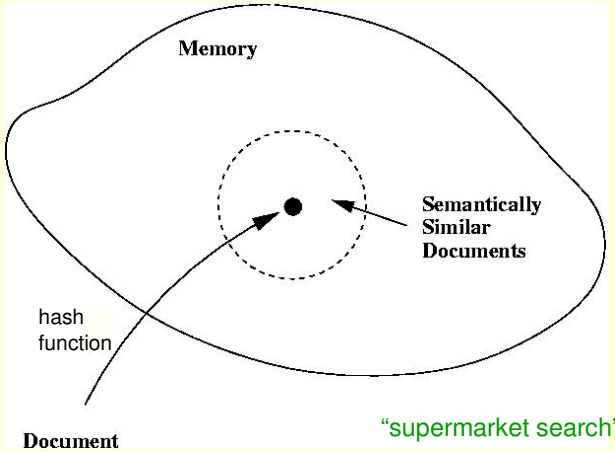


Finding binary codes for documents

- Train an auto-encoder using 30 logistic units for the code layer.
- During the fine-tuning stage, add noise to the inputs to the code units.
  - The "noise" vector for each training case is fixed. So we still get a deterministic gradient.
  - The noise forces their activities to become bimodal in order to resist the effects of the noise.
  - Then we simply round the activities of the 30 code units to 1 or 0.



Using a deep autoencoder as a hash-function for finding approximate matches



"supermarket search"

How good is a shortlist found this way?

- We have only implemented it for a million documents with 20-bit codes --- but what could possibly go wrong?
  - A 20-D hypercube allows us to capture enough of the similarity structure of our document set.
- The shortlist found using binary codes actually improves the precision-recall curves of TF-IDF.
  - Locality sensitive hashing (the fastest other method) is 50 times slower and has worse precision-recall curves.

Time series models

- Inference is difficult in directed models of time series if we use non-linear distributed representations in the hidden units.
  - It is hard to fit Dynamic Bayes Nets to high-dimensional sequences (e.g motion capture data).
- So people tend to avoid distributed representations and use much weaker methods (e.g. HMM's).

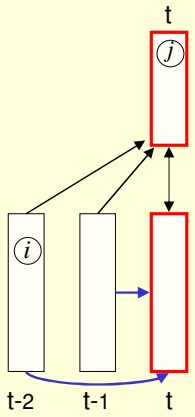
Time series models

- If we really need distributed representations (which we nearly always do), we can make inference much simpler by using three tricks:
  - Use an RBM for the interactions between hidden and visible variables. This ensures that the main source of information wants the posterior to be factorial.
  - Model short-range temporal information by allowing several previous frames to provide input to the hidden units and to the visible units.
- This leads to a temporal module that can be stacked
  - So we can use greedy learning to learn deep models of temporal structure.

The conditional RBM model

- Given the data and the previous hidden state, the hidden units at time t are conditionally independent.
  - So online inference is very easy
- Learning can be done by using contrastive divergence.
  - Reconstruct the data at time t from the inferred states of the hidden units and the earlier states of the visibles.
  - The temporal connections can be learned as if they were additional biases

$$\Delta w_{ij} = s_i \left(< s_j >_{data} - < s_j >_{recon} \right)$$

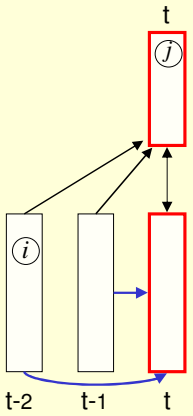


Why the lateral connections do not cause problems

- The lateral connections do not mess up contrastive divergence learning because:
  - We know the initial state of the visible units, so we know the initial effect of the lateral connections.
  - It is not necessary for the reconstructions to be at equilibrium with the hidden units.
  - The important thing for contrastive divergence is to ensure the hiddens are in equilibrium with the visibles whenever statistics are measured.

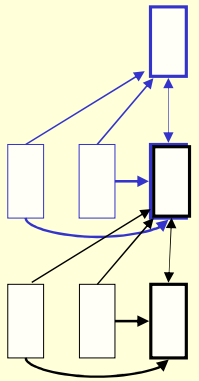
Generating from a learned model

- The inputs from the earlier states of the visible units create dynamic biases for the hidden and current visible units.
- Perform alternating Gibbs sampling for a few iterations between the hidden units and the current visible units.
  - This picks new hidden and visible states that are compatible with each other and with the recent history.



Stacking temporal RBM's

- Treat the hidden activities of the first level TRBM as the data for the second-level TRBM.
  - So when we learn the second level, we get connections across time in the first hidden layer.
- After greedy learning, we can generate from the composite model
  - First, generate from the top-level model by using alternating Gibbs sampling between the current hiddens and visibles of the top-level model, using the dynamic biases created by the previous top-level visibles.
  - Then do a single top-down pass through the lower layers, but using the inputs coming from earlier states of each layer.



An application to modeling motion capture data

- Human motion can be captured by placing reflective markers on the joints and then using lots of infrared cameras to track the 3-D positions of the markers.
- Given a skeletal model, the 3-D positions of the markers can be converted into the joint angles plus 6 parameters that describe the 3-D position and the roll, pitch and yaw of the pelvis.
  - We only represent changes in yaw because physics doesn't care about its value and we want to avoid circular variables.

Modeling multiple types of motion

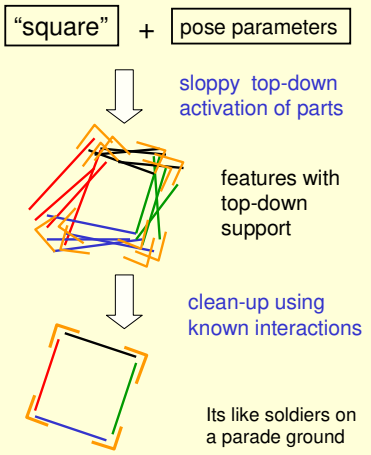
- We can easily learn to model walking and running in a single model.
  - This means we can share a lot of knowledge.
  - It should also make it much easier to learn nice transitions between walking and running.
    - In a switching mixture of dynamical systems its hard to get the latent variables to join up nicely when we switch from one system to another.
- Because we can do online inference (slightly incorrectly), we can fill in missing markers in real time.

Show Graham Taylor's movies

(available at www.cs.toronto/~hinton)

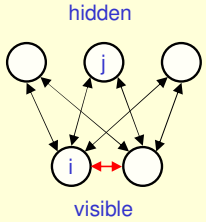
Generating the parts of an object

- One way to maintain the constraints between the parts is to generate each part very accurately
  - But this would require a lot of communication bandwidth.
- Sloppy top-down specification of the parts is less demanding
  - but it messes up relationships between features
  - so use redundant features and use lateral interactions to clean up the mess.
- Each transformed feature helps to locate the others
  - This allows a noisy channel



Semi-restricted Boltzmann Machines

- We restrict the connectivity to make learning easier.
- Contrastive divergence learning requires the hidden units to be in conditional equilibrium with the visibles.
  - But it does not require the visible units to be in conditional equilibrium with the hiddens.
  - All we require is that the visible units are closer to equilibrium in the reconstructions than in the data.
- So we can allow connections between the visibles.



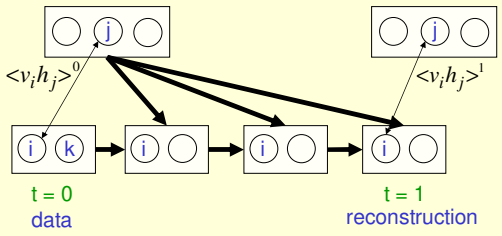
Learning in Semi-restricted Boltzmann Machines

- Method 1: To form a reconstruction, cycle through the visible units updating each in turn using the top-down input from the hiddens plus the lateral input from the other visibles.
- Method 2: Use "mean field" visible units that have real values. Update them all in parallel.
  - Use damping to prevent oscillations

$$p_i^{t+1} = \lambda p_i^t + (1-\lambda)\,\sigma(x_i)$$

damping — total input to i

Learning a semi-restricted Boltzmann Machine



$$\Delta w_{ij} = \varepsilon\,(<v_i h_j>^0 - <v_i h_j>^1)$$

$$\Delta l_{ik} = \varepsilon\,(<v_i v_k>^0 - <v_i v_k>^1)$$

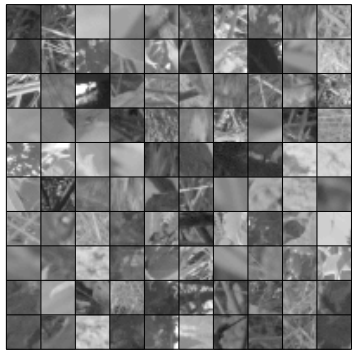
1. Start with a training vector on the visible units.
2. Update all the hidden units in parallel
3. Repeatedly update the all the visible units in parallel using mean-field updates (with the hiddens fixed) to get a "reconstruction".
4. Update the hidden units again.

Results on modeling natural image patches using a stack of RBM's (Osindero and Hinton)

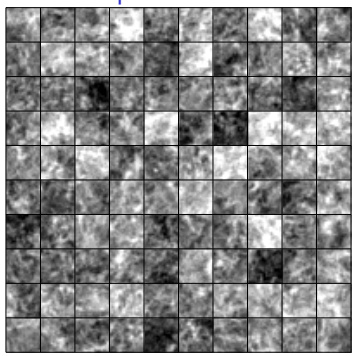
- 100,000 Van Hateren image patches, each 20x20
- Stack of RBM's learned one at a time.
- 400 Gaussian visible units that see whitened image patches.
- 400→2000→500→1000
- Hidden units are all binary with learned lateral connections when they are the visible units of their RBM.
- Generation involves letting the visible units of each RBM settle using mean field with the already decided states in the level above determining the effective biases.

Without lateral connections

real data

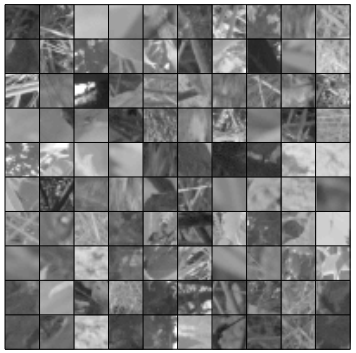


samples from model

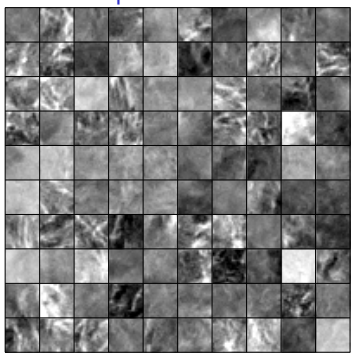


With lateral connections

real data



samples from model



A funny way to use an MRF

- The lateral connections form an MRF.
- The MRF is used during learning and generation.
- The MRF is not used for inference.
  - This is a novel idea so vision researchers don't like it.
- The MRF enforces constraints. During inference, constraints do not need to be enforced because the data obeys them.
  - The constraints only need to be enforced during generation.
- Unobserved hidden units cannot enforce constraints.
  - This requires lateral connections or observed descendants.

Why do we whiten data?

- Images typically have strong pair-wise correlations.
- Learning higher order statistics is difficult when there are strong pair-wise correlations.
  - Small changes in parameter values that improve the modeling of higher order statistics may be rejected because they form a slightly worse model of the much stronger pair-wise statistics.
- So we often remove the second-order statistics before trying to learn the higher-order statistics.

Whitening the learning signal instead of the data

- Contrastive divergence learning can remove the effects of the second-order statistics on the learning without actually changing the data.
  - The lateral connections model the second order statistics
  - If a pixel can be reconstructed correctly using second order statistics, its will be the same in the reconstruction as in the data.
  - The hidden units can then focus on modeling high-order structure that cannot be predicted by the lateral connections.

Higher order Boltzmann machines

- The usual energy function is quadratic in the states:

$$E = bias\ terms\ -\sum_{i<j} s_i s_j\ w_{ij}$$

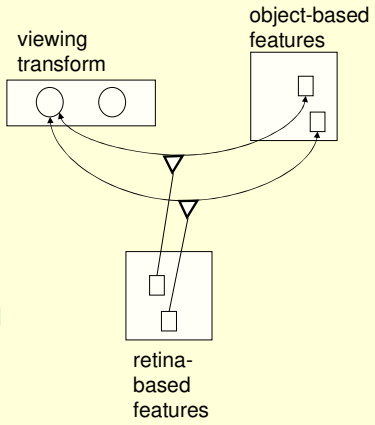
- But we could use higher order interactions:

$$E = bias\ terms\ -\sum_{i<j<k} s_i s_j s_k\ w_{ijk}$$

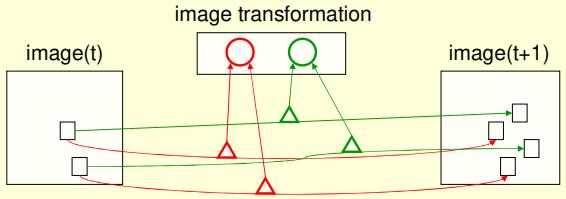
- Unit k acts as a switch. When unit k is on, it switches in the pairwise interaction between unit i and unit j.
  - Units i and j can also be viewed as switches that control the pairwise interactions between j and k or between i and k.

A picture of a conditional, higher-order Boltzmann machine (1985)

- We can view it as a Boltzmann machine in which the inputs create interactions between the other variables.
  - This type of model is now called a conditional random field.



Using conditional higher-order Boltzmann machines to model image transformations

(Memisevic and Hinton, 2007)

- A global transformation specifies which pixel goes to which other pixel.
- Conversely, each pair of similar intensity pixels, one in each image, votes for a particular global transformation.