To: Amanda
From: Daddy
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About: How to solve math problems

There are 4 steps in solving the kind of math problem you showed me. I'll list the steps first, then explain them after. Read the explanations slowly, and make sure you understand them. If anything is unclear, ask me about it. The steps are:

1 Choose your variables.
2 Translate the given information into equations.
3 Solve the equations.
4 Write the answer to the questions asked.

1 Choose your variables.

A math problem talks about some quantities, such as people’s ages, or the lengths of things, or the weights of things, or the numbers of things. Choose a letter for each quantity. Try to choose a letter that suggests the thing it stands for. For example, if the question talks about Alice's age and Mary's age, use a for Alice's age and m for Mary's age. Always write down what you have decided, like this:

Let a be Alice's age.
Let m be Mary's age.

If you choose the letter ℓ , make it look like that so it won't be confused with the number 1 . If you choose the letter O , make it look like that so it won't be confused with the number 0 . If you choose the letter t , make it look like that so it won't be confused with the $+$ sign. If you choose the letter x , make it look like that so it won't be confused with the $\times$ sign.

2 Translate the given information into equations.

Try to make the most direct translation possible to make it easier to see whether you have it right. For example, if it says “Alice is 11 years older than Mary.”, you write $a = 11+m$. The same given information can be written as $a–11 = m$ or as $a–m = 11$ or as $m = a–11$, but $a = 11+m$ is best because it is the most direct translation of “Alice is 11 years older than Mary.”. After you have translated the given information into equations, you put aside the given information; you don't need it anymore.

3 Solve the equations.

To solve equations, you need to know about four things:
3.1 reordering and regrouping
3.2 distributing and factoring
3.3 substituting one side for the other
3.4 doing the same thing to each side

3.1 reordering and regrouping

Additions can be reordered. The law

$$x+y = y+x$$

says that when you add two numbers, it doesn't matter which one is first and which one is second. For example, $2+3$ gives the same answer as $3+2$. Additions can be regrouped. The law

$$(x+y)+z = x+(y+z)$$

says that when you add three numbers, it doesn't matter whether you add the first two and then add the last, or you add the last two and then add the first; either way you get the same answer. When you add lots of numbers, you can reorder them and regroup them any way you want. For example,

$$(1+3)+(5+(7+9)) = (((5+9)+3)+7)+1$$

The left side of the equation says: first add $1+3$, which makes 4 . Then add $7+9$, which makes 16 . Then add $5+16$, which makes 21 . And finally add $4+21$, which makes 25 . The right side says: first add $5+9$, which makes 14 . Then add $14+3$, which makes 17 . Then add $17+7$, which makes 24 . Then add $24+1$, which makes 25 .

Multiplication is like addition in one way: it doesn't matter how you order or group the numbers. For multiplication we have the laws

$$x\,y = y\,x$$

$$(x\,y)\,z = x\,(y\,z)$$

So, for example,

$$2\times((3\times5)\times7) = (5\times3)\times(7\times2)$$

Notice that multiplication can be written with a $\times$ sign or with no sign at all. When we multiply variables like $x\,y$ or a number and a variable like $2x$ we tend not to use a $\times$ sign, but we could if we want to. When we multiply numbers we use a $\times$ sign because we don't want 2×3 to look like twenty-three.

Subtraction and division are not like addition multiplication: it does matter what order and how you group them. $2–3$ is not the same as $3–2$. And $2/3$ is not the same as $3/2$. And $(8–4)–2$ makes 2 , but $8–(4–2)$ makes 6 . So we can't just regroup subtractions. We also can't regroup divisions.

Grouping is not always indicated by brackets. For example, $2+3\times4$ means first multiply 3 by 4 which makes 12 ; the 3×4 is grouped together just as if you wrote $2+(3\times4)$. Then multiply 2 times 12 which makes 24 . Another example is $9–4+3$; first you subtract 9 minus 4 which makes 5 , then you add 5 plus 3 which makes 8 ; you don't add 4 plus 3 . The grouping is $(9–4)+3$, not $9–(4+3)$. Multiplications and divisions are done first, then additions and subtractions, unless parentheses indicate otherwise.

3.2 distributing and factoring

The law

$$x\,(y+z) = x\,y + x\,z$$

says that if you have to add y plus z and then multiply by x , you can instead multiply x by y , and also multiply x and z , and then add the results; that's called distributing. It also says that if you have to multiply x by y , and also multiply x and z , and then add the results, you can instead add y plus z and then multiply by x ; that's called factoring. For example,

$$2\times(3+4) = 2\times3 + 2\times4$$

Each side is 14 .

3.3 substituting one side for the other

Whenever you have an equation, you are allowed to substitute one side for the other. For example, suppose we have two equations
(1) $a+b = 2b$
(2) $b(a+b) = a$
Because we have equation (1), we can replace $a+b$ by $2b$ anywhere we want. Also, we can replace $2b$ by $a+b$ anywhere we want. In equation (2) we see $a+b$, so we can replace it by $2b$ and get equation
(3) $b(2b) = a$
Because we have equation (2) we can replace $b(a+b)$ by a , and we can replace a by $b(a+b)$. There is an a in equation (1), so we can make equation
(4) $b(a+b)+b = 2b$
And so on.

There are three things to be careful of when substituting one side of an equation for the other. First, check the grouping before you substitute. For example, suppose we have the equations
(5) $a+b = 2$
(6) $5–a+b = 3$
Equation (5) says we can replace $a+b$ with 2 . And in equation (6) it looks like there's an $a+b$. But wait! In equation (6) we aren't adding a and b ; we're subtracting 5 minus a , and then adding b . With brackets it would be $(5–a)+b$, not $5–(a+b)$. So we can't replace $a+b$ in (6). (This wouldn't be a problem if we always showed the grouping with brackets.)

Second, check the grouping after you substitute. For example, suppose we have the equations
(7) $a = b+c$
(8) $b = 2a$
Equation (7) says we can replace a with $b+c$, and there's an a in equation (8) which we can replace. But we don't get
(9) $b = 2b+c$
In (8), the a is multiplied by 2 , so in the new equation, whatever we replace a with should be multiplied by 2 . We get
(10) $b = 2(b+c)$
We sometimes have to add brackets to get the grouping right. (This wouldn't be a problem if we always showed the grouping with brackets.)

Third, if you have $2a$ and you are replacing the a with 3 , you get 2×3 ; notice that you have to put in a $\times$ sign so 2 times 3 won't look like 23 . (This wouldn't be a problem if we always used the $\times$ sign.)

3.4 doing the same thing to each side

You can always do the same thing to each side of an equation. For example, if we have the equation
(11) $a+b = 2a$
we can add 6 to each side and get
(12) $a+b+6 = 2a+6$
Or we can subtract the same thing from each side, or multiply each side by the same thing, or divide each side by the same thing. But you have to be careful of the grouping. For example, from (11) we can multiply each side by 2 , and we get
(13) $2(a+b) = 2\times2a$

Suppose we have
(14) $a+b = c$
and we subtract b from each side. Then we get
(15) $a+b-b = c-b$
We can simplify the left side of (15) because if you add b and then subtract b that's the same as doing nothing, so we get
(16) $a = c-b$
Now here's a shortcut. Instead of going from (14) to (15) to (16) we can use a quick trick to get from (14) straight to (16). We just move the $+b$ from the left side to the right side, but in the move the $+$ becomes $-$. Similarly, if we have equation (16), we can move the $-b$ from the right side to the left side, but the $-$ becomes $+$ and we get (14). Really, that's adding b to both sides of (16) to get
(17) $a+b = c-b+b$
and then simplifying the right side to get (14).

Suppose we have
(18) $ab = c$
and we divide both sides by b. Then we get
(19) $ab/b = c/b$
We can simplify the left side of (19) because if you multiply by b and then divide by b that's the same as doing nothing, so we get
(20) $a = c/b$
Now here's a shortcut. Instead of going from (18) to (19) to (20) we can use a quick trick to get from (18) straight to (20). We just move the $\times b$ (ok, the $\times$ isn't there, but ab means $a\times b$) from the left side to the right side, but in the move the $\times$ becomes $/$. Similarly, if we have equation (20), we can move the $/b$ from the right side to the left side, but the $/$ becomes $\times$ and we get (18). Really, that's multiplying both sides of (20) by b to get
(21) $ab = c/b\times b$
and then simplifying the right side to get (18).

In general, we can move things from one side of an equation to the other, but in the move, $+$ becomes $-$, $-$ becomes $+$, $\times$ becomes $/$, and $/$ becomes $\times$.

Solving equations means creating equations that have just a variable on one side and a number on the other, like $x=3$. Usually you need to find an equation like that for each variable, but sometimes the problem needs a solution for only some (not all) of your variables.

Solving equations is like a board game (like chess), except there's only one player. The given information gives you the first few equations; that's like the initial placement of the pieces on the board. Each problem gives you a different initial placement of pieces. Each move of the game gives you a new equation. The legal moves are: reordering and regrouping, distributing and factoring, substituting one side for the other, doing the same thing to each side. If you solve the equations, you win. Since there's only one player, maybe it's more like a puzzle than a game.

4 Write the answer to the questions asked.

Once you have solved the equations, you look at the problem and see what it asked. Then you answer the questions in English. For example, if the problem asks “How old is Alice?”, and your solution has the equation $a=14$, then you say “Alice is 14 years old.”.

Example Problems

Now let's try some problems. You try each problem by yourself without looking at my solution. When you're done or hopelessly stuck, then look at my solution.

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1. Barbara is 142 cm tall. This is 2 cm less than 3 times her height at birth. Find her height at birth.

Let B be Barbara's height now in cm.
Let b be Barbara's height at birth in cm.
The given information is equations (1) and (2).
(1) $B = 142$
(2) $B = 3b - 2$
Use (1) to replace B in (2).
(3) $142 = 3b - 2$
Add 2 to both sides of (3). (Or, move -2 from right side of (3) to left side, but $-$ becomes $+$.)
(4) $142+2 = 3b$
Divide both sides of (4) by 3 . (Or, move 3 from right side of (3) to left side, but it becomes a denominator.)
(5) $(142+2)/3 = b$
Do the arithmetic.
(6) $48 = b$
Barbara was 48 cm at birth.

It would have been all right to introduce just one variable b , and to say that (3) is the given information.

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2. A bicycle is on sale at \$12 more than half of the regular price. If the sale price is \$75, find the regular price.

Let the regular price be r dollars.
Let the sale price be s dollars.
The given information is (1) and (2).
(1) $s = 12 + r/2$
(2) $s = 75$
Use (2) to replace s in (1).
(3) $75 = 12 + r/2$
Subtract 12 from each side of (3).
(4) $75-12 = r/2$
Multiply each side of (4) by 2 .
(5) $(75-12)\times 2 = r$
Do the arithmetic.
(6) $126 = r$
The regular price is \$126.

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3. A set of children's blocks contains three shapes: longs, flats, and cubes. There are 3 times as many longs as cubes, and 30 fewer flats than longs. If there are 600 blocks in all, how many longs are there?

Let the number of longs be ℓ .
Let the number of flats be f .
Let the number of cubes be c .
The given information is (1), (2), and (3).

(1)	$\ell = 3c$	
(2)	$f = \ell - 30$	
(3)	$\ell + f + c = 600$	Use (2) to replace f in (3).
(4)	$\ell + \ell - 30 + c = 600$	Use (1) to replace ℓ (both of them) in (4).
(5)	$3c + 3c - 30 + c = 600$	Add 30 to both sides of (5).
(6)	$3c + 3c + c = 600 + 30$	Use factoring to add up the c's.
(7)	$7c = 600+30$	Divide both sides by 7 .
(8)	$c = (600+30)/7$	Do the arithmetic.
(9)	$c = 90$	Use (9) to replace c in (1).
(10)	$\ell = 3\times 90$	Do the arithmetic.
(11)	$\ell = 270$	

There are 270 longs.

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4. The length of a rectangular field is 7 m less than 4 times the width. The perimeter is 136 m. Find the width and length.

Let the width be w in meters.
Let the length be ℓ in meters.
The given information is (1) and (2).

(1)	$\ell = 4w - 7$	
(2)	$\ell + w + \ell + w = 136$	Use (1) to replace ℓ in (2).
(3)	$4w - 7 + w + 4w - 7 + w = 136$	Add 14 to each side.
(4)	$4w + w + 4w + w = 136+14$	Add up the w's (factoring).
(5)	$10w = 136+14$	Divide each side by 10 .
(6)	$w = (136+14)/10$	Do the arithmetic.
(7)	$w = 15$	Use (7) to replace w in (1).
(8)	$\ell = 4\times 15 - 7$	Do the arithmetic.
(9)	$\ell = 53$	

The width is 15 m and the length is 53 m.

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5. Find three consecutive odd integers such that the sum of the smallest and 7 times the largest is 68.

Let the smallest of the three odd consecutive integers be s .
Then the next one is $s+2$ and the largest one is $s+4$.
The given information is (1).

(1)	$s + 7(s+4) = 68$	Distribute the 7 across $s+4$.
(2)	$s + 7s + 7\times4 = 68$	Subtract 7×4 from both sides.
(3)	$s + 7s = 68 - 7\times4$	Add the s's and do the arithmetic.
(4)	$8s = 40$	Divide both sides by 8 .
(5)	$s = 5$	

The three consecutive integers are 5 , 7 , and 9 .

It would have been all right to choose three variables, one for each integer.

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6. Mark is 11 years older than his sister. In 8 years he will be twice as old as she will be then. How old are they now?

Let Mark's age be m in years.
Let his sister's age be s in years.
The given information is (1) and (2).

(1)	$m = 11+s$	
(2)	$m+8 = 2(s+8)$	Use (1) to replace m in (2).
(3)	$11+s+8 = 2(s+8)$	Distribute the 2 across $s+8$
(4)	$11+s+8 = 2s + 2\times8$	Subtract $s + 2\times8$ from both sides.
(5)	$11 + 8 - 2\times8 = s$	Do the arithmetic.
(6)	$3 = s$	Use (6) to replace s in (1).
(7)	$m = 11+3$	Do the arithmetic.
(8)	$m = 14$	

Mark is 14 and his sister is 3.

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7. A Sugar Dud has 30 less than twice as many calories as a Krunchy Krum. If 5 Sugar Duds have the same number of calories as 8 Krunchy Krums, how many calories are in each?

Let s be the number of calories in a Sugar Dud.
Let k be the number of calories in a Krunchy Krum.
The given information is (1) and (2).

(1)	$s = 2k - 30$	
(2)	$5s = 8k$	Use (1) to replace s in (2).
(3)	$5(2k - 30) = 8k$	Distribute 5 across $2k - 30$.
(4)	$5\times2k - 5\times30 = 8k$	Add 5×30 to each side and subtract $8k$ from each side.
(5)	$5\times2k - 8k = 5\times30$	Subtract the k's on the left, and multiply on the right.
(6)	$2k = 150$	Divide each side by 2 .
(7)	$k = 75$	Use (6) to replace $2k$ in (1).
(8)	$s = 150 - 30$	Do the arithmetic.
(9)	$s = 120$	

There are 120 calories in a Sugar Dud and 75 calories in a Krunchy Krum.

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8. The perimeter of a triangle is 71 cm. The first side is 3 cm shorter than the second side, and the third side is twice as long as the first side. Find the length of the longest side.

Let f be the length of the first side in cm.
Let s be the length of the second side in cm.
Let t be the length of the third side in cm.
The given information is (1), (2), and (3).

(1)	$f+s+t = 71$	
(2)	$f = s-3$	
(3)	$t = 2f$	Use (3) to replace t in (1).
(4)	$f + s + 2f = 71$	Use (2) to replace f in (4).
(5)	$s-3 + s + 2(s-3) = 71$	Distribute 2 across $s-3$
(6)	$s - 3 + s + 2s - 2\times3 = 71$	Move the -3 and -2×3 to the right side
(7)	$s + s + 2s = 71 + 3 + 2\times3$	Do the arithmetic.
(8)	$4s = 80$	Divide both sides by 4
(9)	$s = 20$	Use (9) to replace s in (2)
(10)	$f = 20-3$	Do the arithmetic
(11)	$f = 17$	Use (11) to replace f in (3)
(12)	$t = 2\times17$	Do the arithmetic
(13	$t = 34$	

The longest side is 34 cm.

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9. In a week Mike ran 8 km farther than Bill, while Pete ran 1 km less than 3 times as far as Bill. If Pete ran 15 km farther than Mike, how many kilometers did Bill run?

Let m be how far Mike ran in km.
Let b be how far Bill ran in km.
Let p be how far Pete ran in km.
The given information is (1), (2), and (3).

(1)	$m = 8+b$	
(2)	$p = 3b - 1$	
(3)	$p = 15+m$	Use (1) to replace m in (3)
(4)	$p = 15+8+b$	Use (2) to replace p in (4)
(5)	$3b - 1 = 15+8+b$	Subtract b from both sides. Add 1 to both sides.
(6)	$2b = 24$	Divide both sides by 2
(7)	$b = 12$	

Bill ran 12 km.

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10. The larger of two consecutive integers is 7 greater than twice the smaller. Find the integers.

Let i be the first integer. Then the second integer is $i+1$.
The given information is (1).

(1)	$i+1 = 7 + 2i$	Subtract i from both sides. Subtract 7 from both sides.
(2)	$1-7 = i$	Do the arithmetic
(3)	$-6 = i$	

The integers are -6 and -5 .

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11. A square and a rectangle have the same perimeter. The length of the rectangle is 4 cm less than twice the side of the square, and the width of the rectangle is 6 cm less than the side of the square. Find the perimeter of each figure.

Let s be the side of the square in cm.
Let ℓ be the length of the rectangle in cm.
Let w be the width of the rectangle in cm.
The given information is (1), (2), and (3).

(1)	$4s = 2(\ell+w)$	
(2)	$\ell = 2s - 4$	
(3)	$w = s - 6$	Use (2) to replace ℓ in (1)
(4)	$4s = 2(2s - 4 + w)$	Use (3) to replace w in (4)
(5)	$4s = 2(2s - 4 + s - 6)$	Distribute
(6)	$4s = 4s - 2\times4 + 2s - 2\times6$	Subtract $4s$, add $2\times4 + 2\times6$ to each side.
(7)	$2\times4 + 2\times6 = 2s$	Divide each side by 2
(8)	$10 = s$	

The perimeter of each figure is 4×10 or 40 cm.

In math class, it may seem like the important thing is to get the right answer. Certainly that's one of the important things. But in life, you get asked to solve problems for which people don't already know the answer. There's no teacher to mark them right or wrong. But it's really important to get it right if a bridge is going to be built based on your answers. Even if you're not building bridges, everyone has little problems all the time, like figuring your finances to pay for your kids' expenses, or buying the right number of tiles of various sizes to tile the walls of your bathroom, and so on. And since you can't check the answer before building the bridge or buying the tiles or whatever, you have to be able to check each step in your solution. So the other important thing (besides getting it right) is to write a clear, well presented solution. If you are an architect or engineer or anyone who uses math professionally, you must show your work to other people to have them check it. So presentation is very important. In math class, it may seem like someone who can just see the answer, skipping lots of steps, is really smart. For example, in problem 11 (above), some “smart” kid might just introduce one variable s, and write down equation (5) directly from the given information, and everyone is impressed and says “How did you get that?”. But in a real life job, if other people don't know how you got your answer, then your answer is worthless, and you get fired. So you have to write equations that represent the given information as directly as possible, and you have to explain each step of your solution as clearly as possible. Those comments in between equations, like “Use (2) to replace ℓ in (1)”, are not just me telling you how to solve the problem; they are part of the presentation of the solution. You have to write them too. The presentation of your solution is just as important as the answer.