X7.5 Let A and B be bunches (anything). Define attachment $A\dagger B$ (pronounced "A attach B") with precedence 6 by the axiom

$$A{:}\ B\dagger C \ = \ A{:}\ B \ \vee \ A{:}\ C$$

(a) Prove $A\dagger null = A$ (identity).

(b) Prove $A\dagger A = A$ (idempotence).

(c) Prove $A\dagger B = B\dagger A$ (symmetry).

(d) Prove $A\dagger(B\dagger C) = (A\dagger B)\dagger C$ (associativity).

(e) By finding a counterexample, disprove $A\dagger B = A,B$.

(f) Implement attachment, with proof.

After trying the question, scroll down to a solution.

§ Lemma: $(A: B) = (A: C) \;=\; B{=}C$

Here is an informal, and therefore unsatisfactory, proof.

If $B{=}C$, then $(A: B) = (A: C)$ and $B{=}C$, so left side = right side.

If $B{\neq}C$, then either there is an element x in B but not in C, or there is an element x in C but not in B. Let $A{=}x$. Then $(A: B)$ and $(A: C)$ are unequal, so left side = right side.

(a) Prove $A\dagger null = A$

§	$\top$	axiom
=	$(A: B\dagger C \;=\; A: B \;\vee\; A: C)$	instance $A{\to}A$ and $B{\to}null$ and $C{\to}null$
⇒	$(A: null\dagger null \;=\; A: null \;\vee\; A: null)$	idempotence
=	$(A: null\dagger null \;=\; A: null)$	lemma
=	$(null\dagger null \;=\; null)$	

Now suppose $A{\neq}null$.

	$\top$	axiom
=	$(A: B\dagger C \;=\; A: B \;\vee\; A: C)$	instance $A{\to}B$ and $B{\to}A$ and $C{\to}null$
⇒	$(B: A\dagger null \;=\; B: A \;\vee\; A: null)$	$A{\neq}null$
=	$(B: A\dagger null \;=\; B: A \;\vee\; \bot)$	
=	$(B: A\dagger null \;=\; B: A)$	lemma
=	$(A\dagger null \;=\; A)$	

(b) Prove $A\dagger A = A$

§	$\top$	axiom
=	$(A: B\dagger C \;=\; A: B \;\vee\; A: C)$	instance $A{\to}A$ and $B{\to}A$ and $C{\to}A$
⇒	$(A: A\dagger A \;=\; A: A \;\vee\; A: A)$	reflexive and idempotent
=	$(A: A\dagger A)$	

	$\top$	axiom
=	$(A: B\dagger C \;=\; A: B \;\vee\; A: C)$	instance $A{\to}A\dagger A$ and $B{\to}A$ and $C{\to}A$
⇒	$(A\dagger A: A\dagger A \;=\; A\dagger A: A \;\vee\; A\dagger A: A)$	reflexive and idempotent
=	$(A\dagger A: A)$	

Therefore, by antisymmetry, $A\dagger A = A$.

(c) Prove $A\dagger B = B\dagger A$

§	$\top$	axiom
=	$(A: B\dagger C \;=\; A: B \;\vee\; A: C)$	symmetry
=	$(A: B\dagger C \;=\; A: C \;\vee\; A: B)$	axiom
=	$(A: B\dagger C \;=\; A: C\dagger B)$	instance $A{\to}A\dagger B$ and $B{\to}B$ and $C{\to}A$
=	$(A\dagger B: B\dagger A \;=\; A\dagger B: A\dagger B)$	reflexive
=	$(A\dagger B: B\dagger A)$	instance $A{\to}B$ and $B{\to}A$
⇒	$(B\dagger A: A\dagger B)$	

Therefore, by antisymmetry of the last two lines, $A\dagger B = B\dagger A$.

(d) Prove $A\dagger(B\dagger C) = (A\dagger B)\dagger C$

§	$D: A\dagger(B\dagger C)$	axiom
=	$D: A \;\vee\; D: B\dagger C$	axiom
=	$D: A \;\vee\; D: B \;\vee\; D: C$	axiom
=	$D: A\dagger B \;\vee\; D: C$	axiom
=	$D: (A\dagger B)\dagger C$	

Therefore, by the lemma, $A\dagger(B\dagger C) = (A\dagger B)\dagger C$.

(e) By finding a counterexample, disprove $A\dagger B = A,B$

§ $2,3{:}\ 2\dagger 3 \ = \ 2,3{:}\ 2 \ \vee \ 2,3{:}\ 3 \ = \ \bot\vee\bot \ = \ \bot$

$2,3{:}\ 2,3 \ = \ \top$

(f) Implement attachment, with proof.

§ $A\dagger B \ = \ \sim(\{A\}, \{B\})$ note: $\sim$ does not distribute over ,

Proof

	$A{:}\ B\dagger C$	use implementation
$=$	$A{:}\ \sim(\{B\}, \{C\})$	$(A\in\{B\})=(A{:}B)$ therefore $(A\in B)=(A{:}\sim B)$
$=$	$A \in (\{B\}, \{C\})$	$\{B\}$ and $\{C\}$ are elements
$=$	$A\in\{B\} \ \vee \ A\in\{C\}$	definition of $\in$ twice
$=$	$A{:}\ B \ \vee \ A{:}\ C$	