X3.0 (duality) Prove the duality laws on page 241:

$$\neg\forall v\cdot\ b \;=\; \exists v\cdot\ \neg b$$
$$\neg\exists v\cdot\ b \;=\; \forall v\cdot\ \neg b$$

from the quantifier axioms on page 240:

(0) $\forall v\text{: } null\cdot\ b \;=\; \top$
(1) $\forall v\text{: } x\cdot\ b \;=\; \langle v\text{: } x\cdot\ b\rangle\ x$
(2) $\forall v\text{: } A,B\cdot\ b \;=\; (\forall v\text{: } A\cdot\ b) \wedge (\forall v\text{: } B\cdot\ b)$
(3) $\forall v\text{: } (\S v\text{: } D\cdot\ b)\cdot\ c \;=\; \forall v\text{: } D\cdot\ b \Rightarrow c$

(4) $\exists v\text{: } null\cdot\ b \;=\; \bot$
(5) $\exists v\text{: } x\cdot\ b \;=\; \langle v\text{: } x\cdot\ b\rangle\ x$
(6) $\exists v\text{: } A,B\cdot\ b \;=\; (\exists v\text{: } A\cdot\ b) \vee (\exists v\text{: } B\cdot\ b)$
(7) $\exists v\text{: } (\S v\text{: } D\cdot\ b)\cdot\ c \;=\; \exists v\text{: } D\cdot\ b \wedge c$

After trying the question, scroll down to the solution.

§ I'll prove

$$\neg\forall v: D\cdot b \;=\; \exists v: D\cdot \neg b$$

by structural induction on the structure of the domain D . The ways of forming a domain are: $D = null$, $D = x$ (an element), $D = A,B$ (a union), and $D = \S v: E\cdot c$ (a comprehension). A domain formed as an intersection can also be formed in another way.

	$\neg\forall v: null\cdot b$	use (0)
$=$	$\neg\top$	
$=$	$\bot$	use (4)
$=$	$\exists v: null\cdot \neg b$	

	$\neg\forall v: x\cdot b$	use (1)
$=$	$\neg\,\langle v: x\cdot b\rangle\, x$	this line is: $\neg($ b but replace v by x $)$
$=$	$\langle v: x\cdot \neg b\rangle\, x$	this line is: $(\neg b$ but replace v by x $)$; now use (5)
$=$	$\exists v: x\cdot \neg b$	

	$\neg\forall v: A,B\cdot b$	use (2)
$=$	$\neg((\forall v: A\cdot b) \wedge (\forall v: B\cdot b))$	now apply the basic de Morgan
$=$	$\neg(\forall v: A\cdot b) \vee \neg(\forall v: B\cdot b)$	now a structural induction step, twice
$=$	$(\exists v: A\cdot \neg b) \vee (\exists v: B\cdot \neg b)$	use (6)
$=$	$\exists v: A,B\cdot \neg b$	

	$\neg\forall v: (\S v: E\cdot c)\cdot b$	use (3)
$=$	$\neg(\forall v: E\cdot c \Rightarrow b)$	now a structural induction step
$=$	$\exists v: E\cdot \neg(c \Rightarrow b)$	some binary algebra
$=$	$\exists v: E\cdot c \wedge \neg b$	use (7)
$=$	$\exists v: (\S v: E\cdot c)\cdot \neg b$	

And the other duality law

$$\neg\exists v: D\cdot b \;=\; \forall v: D\cdot \neg b$$

can be proven similarly. Or we can prove it from the one we just proved.

	$\neg\exists v: D\cdot b$	double negation
$=$	$\neg\exists v: D\cdot \neg\neg b$	use what we just proved
$=$	$\neg\neg\forall v: D\cdot \neg b$	double negation
$=$	$\forall v: D\cdot \neg b$	