99 (swapping partners) There is a finite bunch of couples. Each couple consists of a man and a woman. The oldest man and the oldest woman have the same age. If any two couples swap partners, forming two new couples, the younger partners of the two new couples have the same age. Prove that in each couple, the partners have the same age.

After trying the question, scroll down to the solution.

§ The domain will always be the couples, so I will omit it. Label the couples. Let $M\ i$ be the age of the man in couple i. Let $W\ i$ be the age of the woman in couple i. We are given that the oldest man and the oldest woman have the same age

$$\Uparrow M \ = \ \Uparrow W$$

which we write slightly more verbosely as

(a) $\Uparrow i\cdot\ M\ i \ = \ \Uparrow i\cdot\ W\ i$

The other piece of given information is

(b) $\forall i, j\cdot\ (M\ i)\downarrow(W\ j) \ = \ (M\ j)\downarrow(W\ i)$

We must prove $\forall i\cdot\ M\ i = W\ i$. But first, here are some lemmas about $\downarrow$ and $\Uparrow$.

(c) $\quad \forall i\cdot\ M\ i \le \Uparrow j\cdot\ M\ j$ — change local variable

$= \quad \forall i\cdot\ M\ i \le \Uparrow i\cdot\ M\ i$ — extreme law

$= \quad \top$

which proves that any man is younger than or the same age as the oldest man. And so

(d) $\forall i\cdot\ (M\ i)\downarrow(\Uparrow j\cdot\ M\ j) \ = \ M\ i$

Similarly

(e) $\forall i\cdot\ (W\ i)\downarrow(\Uparrow j\cdot\ W\ j) \ = \ W\ i$

Now the desired theorem. Working within the universal quantifier,

$\quad M\ i = W\ i$ — use (d) and (e) and *min* is symmetric

$= \quad (M\ i)\downarrow(\Uparrow j\cdot\ M\ j) \ = \ (\Uparrow j\cdot\ W\ j)\downarrow(W\ i)$ — now use (a) twice

$= \quad (M\ i)\downarrow(\Uparrow j\cdot\ W\ j) \ = \ (\Uparrow j\cdot\ M\ j)\downarrow(W\ i)$ — now distribute twice

$= \quad (\Uparrow j\cdot\ (M\ i)\downarrow(W\ j)) \ = \ (\Uparrow j\cdot\ (M\ j)\downarrow(W\ i))$ — use axiom (b)

$= \quad \top$