All unicorns are white.
All unicorns are black.
No unicorn is both white and black.

Are these statements consistent? What, if anything, can we conclude about unicorns?

§ Let unicorn be all unicorns. Let white and black be predicates on unicorns. Then

All unicorns are white:
(a) \( \forall u: \text{unicorn} \cdot \text{white } u \)

All unicorns are black:
(b) \( \forall u: \text{unicorn} \cdot \text{black } u \)

No unicorn is both white and black:
(c) \( \neg \exists u: \text{unicorn} \cdot \text{white } u \land \text{black } u \)

We take (a), (b), and (c) as axioms. Their conjunction is then a theorem.

\( (\forall u: \text{unicorn} \cdot \text{white } u) \land (\forall u: \text{unicorn} \cdot \text{black } u) \land (\neg \exists u: \text{unicorn} \cdot \text{white } u \land \text{black } u) \)

Using a duality law (deMorgan) on (c), we can change it to a universal quantification:

\( (\forall u: \text{unicorn} \cdot \text{white } u) \land (\forall u: \text{unicorn} \cdot \text{black } u) \land (\forall u: \text{unicorn} \cdot \neg (\text{white } u \land \text{black } u)) \)

Now we can use a splitting law to combine the three main conjuncts

\( \forall u: \text{unicorn} \cdot (\text{white } u \land \text{black } u) \land (\neg (\text{white } u \land \text{black } u)) \) Law of Noncontradiction

Now suppose \( U \) is a particular unicorn, \( U: \text{unicorn} \). Then by Specialization

\( \bot \)

So the statements are inconsistent under the assumption that \( U \) is a unicorn. But if \( \text{unicorn} = \text{null} \), then by the first quantifier law \( (\forall v: \text{null} \cdot b) \), all three axioms are already theorems. If we are given (a), (b), and (c) as axioms, we must conclude that there are no unicorns.