94(a) If P: $bin \to bin$ is monotonic, prove $(\exists x \cdot P\ x) = P\ \top$ and $(\forall x \cdot P\ x) = P\ \bot$.

(b) If P: $bin \to bin$ is antimonotonic, prove $(\exists x \cdot P\ x) = P\ \bot$ and $(\forall x \cdot P\ x) = P\ \top$.

After trying the question, scroll down to the solution.

§(a) $\top$ — definition of monotonic

$=$ $\forall x, y: bin\cdot (x \Rightarrow y \implies P\,x \Rightarrow P\,y)$ — specialize

$\implies$ $(\bot \Rightarrow \top \implies P\,\bot \Rightarrow P\,\top)$ — base and identity laws

$=$ $P\,\bot \Rightarrow P\,\top$ — laws of inclusion

$=$ $(P\,\bot \vee P\,\top) = P\,\top \;\wedge\; (P\,\bot \wedge P\,\top) = P\,\bot$ — quantifier laws: one-element domain

$=$ $((\exists x: \bot\cdot P\,x) \vee (\exists x: \top\cdot P\,x)) = P\,\top \;\wedge\; ((\forall x: \bot\cdot P\,x) \wedge (\forall x: \top\cdot P\,x)) = P\,\bot$

quantifier laws: union domain

$=$ $(\exists x: bin\cdot P\,x) = P\,\top \;\wedge\; (\forall x: bin\cdot P\,x) = P\,\bot$

§(b) $\top$ — definition of antimonotonic

$=$ $\forall x, y: bin\cdot (x \Rightarrow y \implies P\,x \Leftarrow P\,y)$ — specialize: $\bot$ for x, and $\top$ for y

$\implies$ $(\bot \Rightarrow \top \implies P\,\bot \Leftarrow P\,\top)$ — base and identity laws

$=$ $P\,\bot \Leftarrow P\,\top$ — idempotent law

$=$ $(P\,\bot \Leftarrow P\,\top) \;\wedge\; (P\,\bot \Leftarrow P\,\top)$ — laws of inclusion

$=$ $(P\,\bot \vee P\,\top) = P\,\bot \;\wedge\; (P\,\bot \wedge P\,\top) = P\,\top$ — quantifier laws: one-element domain

$=$ $((\exists x: \bot\cdot P\,x) \vee (\exists x: \top\cdot P\,x)) = P\,\bot \;\wedge\; ((\forall x: \bot\cdot P\,x) \wedge (\forall x: \top\cdot P\,x)) = P\,\top$

quantifier laws: union domain

$=$ $(\exists x: bin\cdot P\,x) = P\,\bot \;\wedge\; (\forall x: bin\cdot P\,x) = P\,\top$