93 Prove $((\exists x\cdot\ P\ x) \Rightarrow (\forall x\cdot\ R\ x \Rightarrow Q\ x)) \land (\exists x\cdot\ P\ x \lor Q\ x) \land (\forall x\cdot\ Q\ x \Rightarrow P\ x) \Rightarrow (\forall x\cdot\ R\ x \Rightarrow P\ x)$

After trying the question, scroll down to the solution.

§ $((\exists x\cdot P\,x) \Rightarrow (\forall x\cdot R\,x \Rightarrow Q\,x)) \wedge (\exists x\cdot P\,x \vee Q\,x) \wedge (\forall x\cdot Q\,x \Rightarrow P\,x)$ idempotence

$= \quad ((\exists x\cdot P\,x) \Rightarrow (\forall x\cdot R\,x \Rightarrow Q\,x)) \wedge (\exists x\cdot P\,x \vee Q\,x) \wedge (\forall x\cdot Q\,x \Rightarrow P\,x) \wedge (\forall x\cdot Q\,x \Rightarrow P\,x)$

distribution

$= \quad ((\exists x\cdot P\,x) \Rightarrow (\forall x\cdot R\,x \Rightarrow Q\,x)) \wedge (\exists x\cdot (P\,x \vee Q\,x) \wedge (\forall x\cdot Q\,x \Rightarrow P\,x)) \wedge (\forall x\cdot Q\,x \Rightarrow P\,x)$

specialization

$= \quad ((\exists x\cdot P\,x) \Rightarrow (\forall x\cdot R\,x \Rightarrow Q\,x)) \wedge (\exists x\cdot (P\,x \vee Q\,x) \wedge (Q\,x \Rightarrow P\,x)) \wedge (\forall x\cdot Q\,x \Rightarrow P\,x)$

use implication $Q\,x \Rightarrow P\,x$ to weaken $Q\,x$, then drop implication

$\Rightarrow \quad ((\exists x\cdot P\,x) \Rightarrow (\forall x\cdot R\,x \Rightarrow Q\,x)) \wedge (\exists x\cdot P\,x \vee P\,x) \wedge (\forall x\cdot Q\,x \Rightarrow P\,x)$ idempotence

$= \quad ((\exists x\cdot P\,x) \Rightarrow (\forall x\cdot R\,x \Rightarrow Q\,x)) \wedge (\exists x\cdot P\,x) \wedge (\forall x\cdot Q\,x \Rightarrow P\,x)$ modus ponens

$\Rightarrow \quad (\forall x\cdot R\,x \Rightarrow Q\,x) \wedge (\forall x\cdot Q\,x \Rightarrow P\,x)$ splitting

$= \quad \forall x\cdot (R\,x \Rightarrow Q\,x) \wedge (Q\,x \Rightarrow P\,x)$ transitivity

$\Rightarrow \quad \forall x\cdot R\,x \Rightarrow P\,x$