Prove that the square of an odd natural number is odd, and the square of an even natural number is even.

The even naturals are $2 \times \text{nat}$ and the odd naturals are $2 \times \text{nat} + 1$. To say that the square of an odd natural is odd is easy:

$$(2 \times \text{nat} + 1)^2 = 2 \times \text{nat} + 1$$

But arithmetic on bunches is tricky; for example, $\text{nat}^2$ and $\text{nat} \times \text{nat}$ differ; $2 \times \text{nat}$ and $\text{nat} + \text{nat}$ differ. To be safe, we prove

$$\forall a : 2 \times \text{nat} + 1 \cdot \exists b : 2 \times \text{nat} + 1 \cdot a^2 = b$$

change of variable, twice

$$\equiv \forall n : \text{nat} \cdot \exists m : \text{nat} \cdot (2 \times n + 1)^2 = 2 \times m + 1$$

various number laws

$$\equiv \forall n : \text{nat} \cdot \exists m : \text{nat} \cdot 2 \times (2 \times n^2 + 2 \times n) + 1 = 2 \times m + 1$$

generalization

$$\iff \forall n : \text{nat} \cdot 2 \times (2 \times n^2 + 2 \times n) + 1 = 2 \times (2 \times n^2 + 2 \times n) + 1$$

reflexivity and identity laws

$$\equiv T$$

To prove the square of an even natural number is even, we prove

$$\forall a : 2 \times \text{nat} \cdot \exists b : 2 \times \text{nat} \cdot a^2 = b$$

change of variable, twice

$$\equiv \forall n : \text{nat} \cdot \exists m : \text{nat} \cdot (2 \times n)^2 = 2 \times m$$

various number laws

$$\equiv \forall n : \text{nat} \cdot \exists m : \text{nat} \cdot 2 \times (2 \times n^2) = 2 \times m$$

generalization

$$\iff \forall n : \text{nat} \cdot 2 \times (2 \times n^2) = 2 \times (2 \times n^2)$$

reflexivity and identity laws

$$\equiv T$$