There is a finite bunch of couples. Each couple consists of a man and a woman. The oldest man and the oldest woman have the same age. If any two couples swap partners, forming two new couples, the younger partners of the two new couples have the same age. Prove that in each couple, the partners have the same age.

The domain will always be the couples, so I will omit it. Label the couples. Let $M_i$ be the age of the man in couple $i$. Let $W_i$ be the age of the woman in couple $i$. We are given that the oldest man and the oldest woman have the same age

$$\text{MAX } M = \text{MAX } W$$

which we write slightly more verbosely as

(a) $\text{MAX } i \cdot M_i = \text{MAX } i \cdot W_i$

The other piece of given information is

(b) $\forall i, j \cdot \text{min } (M_i)(W_j) = \text{min } (M_j)(W_i)$

We must prove $\forall i \cdot M_i = W_i$. But first, here are some lemmas about $\text{min}$ and $\text{MAX}$.

(c) $\forall i \cdot M_i \leq \text{MAX } j \cdot M_j$

$$\equiv \forall i \cdot M_i \leq \text{MAX } i \cdot M_i$$

$$\equiv \top$$

which proves that any man is younger than or the same age as the oldest man. And so

(d) $\forall i \cdot \text{min } (M_i)(\text{MAX } j \cdot M_j) = M_i$

Similarly

(e) $\forall i \cdot \text{min } (W_i)(\text{MAX } j \cdot W_j) = W_i$

Now the desired theorem. Working within the universal quantifier,

$$M_i = W_i$$

use (d) and (e) and $\text{min}$ is symmetric

$$\equiv \text{min } (M_i)(\text{MAX } j \cdot M_j) = \text{min } (\text{MAX } j \cdot W_j)(W_i)$$

now use (a) twice

$$\equiv \text{min } (M_i)(\text{MAX } j \cdot W_j) = \text{min } (\text{MAX } j \cdot M_j)(W_i)$$

now distribute twice

$$\equiv (\text{MAX } j \cdot \text{min } (M_i)(W_j)) = (\text{MAX } j \cdot \text{min } (M_j)(W_i))$$

use axiom (b)

$$\equiv \top$$