9 Consider a fully bracketed expression containing only the symbols $\top$ $\perp = \neq ($ $)$ in any quantity and any syntactically acceptable order.

(a) Show that all syntactically acceptable rearrangements are equivalent.

(b) Show that it is equivalent to any expression obtained from it by making an even number of the following substitutions: $\top$ for $\perp$, $\perp$ for $\top$, $=$ for $\neq$, $\neq$ for $=$.

After trying the question, scroll down to the solution.

§ The proofs will be by induction over the structure of the expressions. Every fully parenthesized expression containing only the symbols $\top$ $\bot$ $=$ $\ne$ ( ) has one of the following four forms: $\top$, $\bot$, $(a=b)$, $(a \ne b)$, where a and b are fully parenthesized expression containing only the symbols $\top$ $\bot$ $=$ $\ne$ ( ) .

(a) Show that all syntactically acceptable rearrangements are equivalent.

§ There are four alternatives. The first two alternatives are just a single symbol, so there are no rearrangements, so all zero rearrangements are equivalent. That's the base case. Now for the induction step.

Suppose the expression is $(a=b)$ for some expressions a and b . The ways of rearranging $(a=b)$ are:
(a) rearrange a
(b) rearrange b
(c) change $(a=b)$ to $(b=a)$
First, consider (a). Make the inductive hypothesis that rearranging a results in an expression that is equivalent to a . Then any expression with subexpression a is equivalent to the same expression with subexpression a replaced by its rearrangement. (This is formalized as the generic law of transparency.) Similarly for (b). For (c), we have the generic law of symmetry of $=$. That completes the proof for expressions of the form $(a=b)$.

Finally, suppose the expression is $(a \ne b)$ for some expressions a and b . The ways of rearranging $(a \ne b)$ are:
(a) rearrange a
(b) rearrange b
(c) change $(a \ne b)$ to $(b \ne a)$
First, consider (a). Make the inductive hypothesis that rearranging a results in an expression that is equivalent to a . Then any expression with subexpression a is equivalent to the same expression with subexpression a replaced by its rearrangement. (This is formalized as the generic law of transparency.) Similarly for (b). For (c), we have the generic law of symmetry of $\ne$.

That completes the proof

(b) Show that it is equivalent to any expression obtained from it by making an even number of the following substitutions: $\top$ for $\bot$, $\bot$ for $\top$, $=$ for $\ne$, $\ne$ for $=$.

§ Zero substitutions means the same expression, which is obviously equivalent. I will show that by making a single one of those substitutions, the expression is negated. Therefore two substitutions are a double negation, which is an equivalent expression. And so on for more substitutions.

If the expression is $\top$, the only substitution is $\bot$ for $\top$, and $\bot$ is the negation of $\top$.

If the expression is $\bot$, the only substitution is $\top$ for $\bot$, and $\top$ is the negation of $\bot$.

Suppose the expression is $(a=b)$ for some expressions a and b . The ways of making one substitution in $(a=b)$ are:
(i) make one substitution in a

(ii) make one substitution in b
(iii) change $(a{=}b)$ to $(a{\ne}b)$
First, consider (i). Make the inductive hypothesis that one substitutions in a negates a, resulting in an expression equivalent to $(\neg a{=}b)$.

$(\neg a{=}b)$ exclusion
$=$ $(a{\ne}b)$ generic unequality
$=$ $\neg(a{=}b)$

so making one substitution in a negates $(a{=}b)$. Similarly for (ii). For (iii),

$(a{\ne}b)$ generic unequality
$=$ $\neg(a{=}b)$

That completes the proof for expressions of the form $(a{=}b)$. Finally, suppose the expression is $(a{\ne}b)$ for some expressions a and b. The ways of making one substitution in $(a{\ne}b)$ are:
(i) make one substitution in a
(ii) make one substitution in b
(iii) change $(a{\ne}b)$ to $(a{=}b)$
First, consider (i). Make the inductive hypothesis that one substitutions in a negates a, resulting in an expression equivalent to $(\neg a{\ne}b)$.

$(\neg a{\ne}b)$ generic unequalty
$=$ $\neg(\neg a{=}b)$ exclusion
$=$ $\neg(a{\ne}b)$

so making one substitution in a negates $(a{\ne}b)$. Similarly for (ii). For (iii),

$(a{=}b)$ double negation
$=$ $\neg\neg(a{=}b)$ generic unequality
$=$ $\neg(a{\ne}b)$

That completes the proof