Express formally that

This exercise illustrates the need for formalization. Even carefully worded informal specifications can be misunderstood.

(a) natural \( n \) is the largest proper (neither 1 nor \( m \)) factor of natural \( m \).

Define predicate \( p \) so that \( px \) means that \( x \) is a proper factor of \( m \).

\[ p = \langle x: \text{nat} \to m: x \times \text{nat} \land x + 1 \land x + m \rangle \]

Then the expression we want is 

\[ pn \land \forall x: \text{nat}: p x \Rightarrow x \leq n \]

or equivalently

\[ n = \text{MAX} x: (\$p): x \]

(b) \( g \) is the greatest common divisor of naturals \( a \) and \( b \).

\[ a, b: g \times \text{nat} \land \neg \exists h: \text{nat}: a, b: h \times \text{nat} \land h \geq g \]

(c) \( m \) is the lowest common multiple of naturals \( a \) and \( b \).

Is 0 a multiple of \( a \) and \( b \)? If so, the answer is \( m = 0 \). If not, then

\[ m: (\text{nat} + 1) \times a \times (\text{nat} + 1) \times b \land \forall z: (\text{nat} + 1) \times a \times (\text{nat} + 1) \times b \times m \leq z \]

or

\[ m = \text{MIN} m: (\text{nat} + 1) \times a \times (\text{nat} + 1) \times b \times m \]

(d) \( p \) is a prime number.

Here are some possible answers.

\[ \forall n: \text{nat}: p: n \times \text{nat} \Rightarrow n = 1 \lor n = p \] says 0 isn't prime and 1 is

\[ \forall n: 0, \ldots, p: n \times \text{nat} \Rightarrow n = 1 \] says 0 and 1 are prime

\[ \notin \forall n: 0, \ldots, p: n \times \text{nat} = 1 \] says 0 and 1 aren't prime

\[ \notin \forall n: 0, \ldots, p: n \times \text{nat} = 2 \] says 0 and 1 aren't prime

\[ p: \text{nat} + 2 \land \neg (p: (\text{nat} + 2) \times (\text{nat} + 2)) \] says 0 and 1 aren't prime

These answers agree on whether numbers in \( \text{nat} + 2 \) are prime, but they disagree on 0 and 1. Most mathematicians want to exclude 1, and they haven't thought about 0.

(e) \( n \) and \( m \) are relatively prime numbers.

\[ \forall f: \text{nat}: n, m: f \times \text{nat} \Rightarrow f = 1 \]

or \[ \notin \forall f: \text{nat}: n, m: f \times \text{nat} = 1 \]

(f) there is at least one and at most a finite number of naturals satisfying predicate \( p \).

\[ 1 \leq \notin (\forall n: \text{nat}: \text{pn}) < \infty \]

and if \( \Box p = \text{nat} \),

\[ 1 \leq \notin (\forall p) < \infty \]

(g) there is no smallest integer.

\[ \neg \exists i: \text{int} \lor \forall j: \text{int} : i \leq j \]

OR \[ \forall i: \text{int} \lor \exists j: \text{int} : j < i \]

(h) between every two rational numbers there is another rational number.

\[ \forall x, z: \text{rat} : x < z \Rightarrow \exists y: \text{rat} : x < y < z \]

One could argue that it should be

\[ \forall x, z: \text{rat} \lor \exists y: \text{rat} : x < y < z \lor z < y < x \]

because it doesn't say “between every two different rational numbers”, or one could argue that “two rational numbers” means “two different rational numbers”.

(i) list \( L \) is a longest segment of list \( M \) that does not contain item \( x \).
§ Let the type of item be \( T \). Define
\[
P = (A: [*T] \to \exists i, j: 0,..,\#M+1 \cdot i \leq j \land A = M[i,..,j] \land \neg x: M(i,..,j))
\]
so that \( PA \) means “ \( A \) is a segment of \( M \) that does not contain item \( x \)”. Now the answer is
\[
PL \land \neg \exists A: [*T] \cdot PA \land \#A > \#L
\]

(j) the segment of list \( L \) from (including) index \( i \) up to (excluding) index \( j \) is a segment whose sum is smallest.

§
\[
0 \leq i \leq j \leq \#L \land \forall x: 0,..,\#L+1 \cdot \forall y: x,..,\#L+1 \cdot (\Sigma L[i,..,j]) \leq (\Sigma L[x,..,y])
\]

(k) \( a \) and \( b \) are items of lists \( A \) and \( B \) (respectively) whose absolute difference is least.

§
\[
a: A(0,..,\#A) \land b: B(0,..,\#B)
\land \neg (\exists c: A(0,..,\#A) \cdot \exists d: B(0,..,\#B) \cdot \text{abs}(c - d) < \text{abs}(a - b))
\lor (\exists i: 0,..,\#A \cdot \exists j: 0,..,\#B \cdot Ai=a \land Bj=b)
\land \neg (\exists i: 0,..,\#A \cdot \exists j: 0,..,\#B \cdot \text{abs}(Ai - Bj) < \text{abs}(a - b))
\]

(l) \( p \) is the length of a longest plateau (segment of equal items) in a non-empty sorted list \( L \).

§ Define
\[
P = (p: 1,..,\#L+1 \to \exists i: 0,..,\#L+1-p \cdot Li=L(i+p-1))
\]
so that \( PP \) says that \( p \) is the length of a plateau in a non-empty sorted list \( L \). Now the answer is
\[
P + 1 \land \neg PP (p+1)
\]

(m) all items that occur in list \( L \) occur in a segment of length 10.

§ It doesn't say the segment must be in list \( L \), but surely that is intended. Must an item occur as many times in the segment as in the list? Probably not. Can it be a different segment for each item? If so, it's trivially true for any list of length at least 10. It must mean that in list \( L \) there is a segment of length 10 containing all items that occur anywhere in list \( L \).
\[
\#L \geq 10 \land \exists i: 0,..,\#L-9 \cdot \forall j: 0,..,\#L \cdot \exists k: i,..,i+10 \cdot Lj=Lk
\]
\lor
\[
\#L \geq 10 \land \exists i: 0,..,\#L-9 \cdot L(0,..,\#L): L(i,..,i+10)
\]

(n) all items of list \( L \) are different (no two items are equal).

§
\[
\neg \exists i, j: 0,..,\#L \cdot i \neq j \land Li=Lj
\]
or
\[
\forall L(0,..,\#L) = \#L
\]

(o) at most one item in list \( L \) occurs more than once.

§
\[
\neg \exists i, j, k, l: 0,..,\#L \cdot i \neq j \land k \neq l \land Li=Lj+Lk=Li
\]

(p) the maximum item in list \( L \) occurs \( m \) times.

§ If the question means exactly \( m \) times, then the answer is
\[
\#L = (\text{MAX } L) = m
\]
If it means at least \( m \) times, then the answer is
\[
\#L = (\text{MAX } L) \geq m
\]

(q) list \( L \) is a permutation of list \( M \).

§
\[
\forall x: (\#L \cdot Li=x) = (\#M \cdot Mi=x)
\]