Express formally that

§ This exercise illustrates the need for formalization. Even carefully worded informal specifications can be misunderstood.

(a) natural $n$ is the largest proper (neither 1 nor $m$) factor of natural $m$.
§ Define predicate $p$ so that $p x$ means that $x$ is a proper factor of $m$.
\[
p = \langle x: \text{nat} \rightarrow m: x \times \text{nat} \land x \neq 1 \land x + m \rangle
\]
Then the expression we want is
\[
p n \land \forall x: \text{nat} p x \Rightarrow x \leq n
\]
or equivalently
\[
n = \text{MAX } x: (p x)
\]

(b) $g$ is the greatest common divisor of naturals $a$ and $b$.
§ $g = \langle a, b: \text{nat} \times \text{nat} \land \neg \exists h: \text{nat} \times \text{nat} \land h > g \rangle$

(c) $m$ is the lowest common multiple of naturals $a$ and $b$.
§ Is 0 a multiple of $a$ and $b$? If so, the answer is $m = 0$. If not, then
\[
m = \text{MIN } m: (n: \text{nat} + 2) \\
\land a \times (n + 1) \land b \times (n + 1) \land m \leq \forall z: (n: \text{nat} + 1) \times a \times (n + 1) \land b \times m
\]
or
\[
m = \text{MIN } m: (n: \text{nat} + 2) \\
\land a \times (n + 1) \land b \times m
\]

(d) $p$ is a prime number.
§ Here are three possible answers.
\[
\forall n: \text{nat} \land p \times n = 1 \lor n = p
\]
\[
\forall n: 0..p \times n = 1
\]
\[
\varphi \exists n: 0..p \times n = 1
\]
These three answers agree on whether numbers in $\text{nat} + 2$ are prime, but they disagree on 0 and 1. The first two say that 0 and 1 are prime and the last one says they aren't. Most mathematicians want to exclude 1, and they haven't thought about 0.

(e) $n$ and $m$ are relatively prime numbers.
§ $\forall f: \text{nat} \times n \land m = f \times \text{nat} \Rightarrow f = 1$
or $\varphi \exists f: \text{nat} \times n \land m = f \times \text{nat} = 1$

(f) there is at least one and at most a finite number of naturals satisfying predicate $p$.
§ $1 \leq \varphi (\exists n: \text{nat} \land p n) < \infty$
and if $\Box p = \text{nat}$
\[
1 \leq \varphi (\exists p) < \infty
\]

(g) there is no smallest integer.
§ $\neg \exists i: \text{int} \land \forall j: \text{int} \times i \leq j$

(h) between every two rational numbers there is another rational number.
§ $\forall x, z: \text{rat} \times x < z \Rightarrow \exists y: \text{rat} \times x < y < z$
One could argue that it should be
\[
\forall x, z: \text{rat} \\
\exists y: \text{rat} \times x < y < z \lor z < y < x
\]
because it doesn't say “between every two different rational numbers”, or one could argue that “two rational numbers” means “two different rational numbers”.

(i) list $L$ is a longest segment of list $M$ that does not contain item $x$.
§ Let the type of item be $T$. Define
\[
P = \langle A: [^* T] \rightarrow \exists i, j: 0..#M + 1 \times i \leq j \land A = M[i..j] \land \neg x: M(i..j) \rangle
\]
so that \( P A \) means “\( A \) is a segment of \( M \) that does not contain item \( x \)”. Now the answer is

\[
P L \land \neg \exists A: [*T] \cdot P A \land #A > #L
\]

(j) the segment of list \( L \) from (including) index \( i \) up to (excluding) index \( j \) is a segment whose sum is smallest.

\[
0 \leq i \leq j \leq #L \land \forall x: 0..#L+1 \cdot \forall y: x..#L+1 \cdot (\sum L[i..j]) \leq (\sum L[x..y])
\]

(k) \( a \) and \( b \) are items of lists \( A \) and \( B \) (respectively) whose absolute difference is least.

\[
\forall x, y: 0..#L+1 \cdot (\sum L[i..j]) \leq (\sum L[x..y])
\]

(l) \( p \) is the length of a longest plateau (segment of equal items) in a non-empty sorted list \( L \).

\[
P = \{ p: 1..#L+1 \rightarrow \exists i: 0..#L+1-p \cdot Li = L(i+p-1) \}
\]

(m) all items that occur in list \( L \) occur in a segment of length 10.

\[
\#L \geq 10 \land \exists i: 0..#L-9 \cdot \forall j: 0..#L \cdot \exists k: i..i+10 \cdot Lj = Lk
\]

(n) all items of list \( L \) are different (no two items are equal).

\[
\neg \exists i, j: 0..#L \cdot i < j \land Li = Lj
\]

(o) at most one item in list \( L \) occurs more than once.

\[
\neg \exists i,j,k,l: 0..#L \cdot i < j \land k < l \land Li = Lj = Lk = Ll
\]

(p) the maximum item in list \( L \) occurs \( m \) times.

\[
\forall x \cdot (\exists i: 0..#L \cdot Li = x) = (\exists i: 0..#M \cdot Mi = x)
\]