7 Prove each of the following laws of Binary Theory using the proof format given in Subsection 1.0.1, and any laws listed in Section 11.3. Do not use the Completion Rule.

(a) **if** a **then** a **else** $\neg a$ **fi**

(b) **if** b **then** c **else** $\neg c$ **fi** = **if** c **then** b **else** $\neg b$ **fi**

(c) **if** $b \wedge c$ **then** P **else** Q **fi** = **if** b **then if** c **then** P **else** Q **fi else** Q **fi**

(d) **if** $b \vee c$ **then** P **else** Q **fi** = **if** b **then** P **else if** c **then** P **else** Q **fi fi**

(e) **if** b **then** P **else if** b **then** Q **else** R **fi fi** = **if** b **then** P **else** R **fi**

(f) **if if** b **then** c **else** d **fi then** P **else** Q **fi**
= **if** b **then if** c **then** P **else** Q **fi else if** d **then** P **else** Q **fi fi**

(g) **if** b **then if** c **then** P **else** R **fi else if** c **then** Q **else** R **fi fi**
= **if** c **then if** b **then** P **else** Q **fi else** R **fi**

(h) **if** b **then if** c **then** P **else** R **fi else if** d **then** Q **else** R **fi fi**
= **if if** b **then** c **else** d **fi then if** b **then** P **else** Q **fi else** R **fi**

After trying the question, scroll down to the solution.

§(a)		$\textbf{if } a \textbf{ then } a \textbf{ else } \neg a \textbf{ fi}$	one-case
	$=$	$a=a$	reflexive
	$=$	$\top$	

Here is another solution.

		$\textbf{if } a \textbf{ then } a \textbf{ else } \neg a \textbf{ fi}$	case analysis
	$=$	$a \land a \;\lor\; \neg a \land \neg a$	idempotence twice
	$=$	$a \lor \neg a$	excluded middle
	$=$	$\top$	

Here is another solution.

		$\textbf{if } a \textbf{ then } a \textbf{ else } \neg a \textbf{ fi}$	context
	$=$	$\textbf{if } a \textbf{ then } \top \textbf{ else } \neg\bot \textbf{ fi}$	binary law
	$=$	$\textbf{if } a \textbf{ then } \top \textbf{ else } \top \textbf{ fi}$	generic case idempotent law
	$=$	$\top$	

§(b)		$\textbf{if } b \textbf{ then } c \textbf{ else } \neg c \textbf{ fi}$	case analysis
	$=$	$b\land c \;\lor\; \neg b\land\neg c$	symmetry twice
	$=$	$c\land b \;\lor\; \neg c\land\neg b$	case analysis
	$=$	$\textbf{if } c \textbf{ then } b \textbf{ else } \neg b \textbf{ fi}$	

§(c)		$\textbf{if } b \textbf{ then if } c \textbf{ then } P \textbf{ else } Q \textbf{ fi else } Q \textbf{ fi}$	case analysis, twice
	$=$	$\underline{b \land (c\land P \lor \neg c\land Q)} \lor \neg b\land Q$	distribution
	$=$	$b\land c\land P \lor \underline{b\land\neg c\land Q \lor \neg b\land Q}$	distribution
	$=$	$b\land c\land P \lor (\underline{b\land\neg c \lor \neg b}) \land Q$	symmetry
	$=$	$b\land c\land P \lor \underline{(\neg b \lor b\land\neg c)} \land Q$	distribution
	$=$	$b\land c\land P \lor \underline{(\neg b \lor b)}\land\underline{(\neg b \lor \neg c)}\land Q$	excluded middle, duality
	$=$	$b\land c\land P \lor \underline{\top\land}\neg(b\land c)\land Q$	identity
	$=$	$b\land c\land P \lor \neg(b\land c)\land Q$	case analysis
	$=$	$\textbf{if } b\land c \textbf{ then } P \textbf{ else } Q \textbf{ fi}$	

§(d)		$\textbf{if } b \textbf{ then } P \textbf{ else if } c \textbf{ then } P \textbf{ else } Q \textbf{ fi fi}$	case analysis twice
	$=$	$b\land P \lor \underline{\neg b\land(c\land P \lor \neg c\land Q)}$	distribute
	$=$	$\underline{b\land P \lor \neg b\land c\land P} \lor \neg b\land\neg c\land Q$	factor (undistribute)
	$=$	$(\underline{b \lor \neg b\land c})\land P \lor \underline{\neg b\land\neg c}\land Q$	distribute, duality
	$=$	$\underline{(b \lor \neg b)\land}(b \lor c)\land P \lor \neg(b \lor c)\land Q$	excluded middle and identity
	$=$	$(b \lor c)\land P \lor \neg(b \lor c)\land Q$	case analysis
	$=$	$\textbf{if } b\lor c \textbf{ then } P \textbf{ else } Q \textbf{ fi}$	

§(e)		$\textbf{if } b \textbf{ then } P \textbf{ else if } b \textbf{ then } Q \textbf{ else } R \textbf{ fi fi}$	context
	$=$	$\textbf{if } b \textbf{ then } P \textbf{ else if } \bot \textbf{ then } Q \textbf{ else } R \textbf{ fi fi}$	case base
	$=$	$\textbf{if } b \textbf{ then } P \textbf{ else } R \textbf{ fi}$	

§(f)		$\textbf{if if } b \textbf{ then } c \textbf{ else } d \textbf{ fi then } P \textbf{ else } Q \textbf{ fi}$	case analysis
	$=$	$\textbf{if } b \textbf{ then } c \textbf{ else } d \textbf{ fi} \land P \;\lor\; \neg\textbf{if } b \textbf{ then } c \textbf{ else } d \textbf{ fi} \land Q$	distribute
	$=$	$\textbf{if } b \textbf{ then } c\land P \textbf{ else } d\land P \textbf{ fi} \;\lor\; \textbf{if } b \textbf{ then } \neg c\land Q \textbf{ else } \neg d\land Q \textbf{ fi}$	distribute
	$=$	$\textbf{if } b \textbf{ then } c\land P \lor \neg c\land Q \textbf{ else } d\land P \lor \neg d\land Q \textbf{ fi}$	case analysis
	$=$	$\textbf{if } b \textbf{ then if } c \textbf{ then } P \textbf{ else } Q \textbf{ fi else if } d \textbf{ then } P \textbf{ else } Q \textbf{ fi fi}$	

§(g) **if** b **then if** c **then** P **else** R **fi else if** c **then** Q **else** R **fi fi** — case idempotent

= **if** c **then if** b **then if** c **then** P **else** R **fi else if** c **then** Q **else** R **fi**
else if b **then if** c **then** P **else** R **fi else if** c **then** Q **else** R **fi fi fi** — context

= **if** c **then if** b **then if** $\top$ **then** P **else** R **fi else if** $\top$ **then** Q **else** R **fi**
else if b **then if** $\bot$ **then** P **else** R **fi else if** $\bot$ **then** Q **else** R **fi fi fi** — case base

= **if** c **then if** b **then** P **else** Q **fi else if** b **then** R **else** R **fi fi** — case idempotent

= **if** c **then if** b **then** P **else** Q **fi else** R **fi**

§(h) **if if** b **then** c **else** d **fi then if** b **then** P **else** Q **fi else** R **fi** — case analysis law

= **if** b **then** c **else** d **fi** $\wedge$ **if** b **then** P **else** Q **fi** $\vee$ $\neg$**if** b **then** c **else** d **fi** $\wedge R$ — four case distributive laws

= **if** b **then** $c \wedge P \vee \neg c \wedge R$ **else** $d \wedge Q \vee \neg d \wedge R$ **fi** — case analysis law twice

= **if** b **then if** c **then** P **else** R **fi else if** d **then** Q **else** R **fi fi**