6 Prove each of the following laws of Binary Theory using the proof format given in Subsection 1.0.1, and any laws listed in Section 11.3. Do not use the Completion Rule.

(a) $a \wedge b \Rightarrow a \vee b$

(b) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \;=\; (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

(c) $\neg a \Rightarrow (a \Rightarrow b)$

(d) $a{=}(b \Rightarrow a) \;=\; a \vee b$

(e) $a{=}(a \Rightarrow b) \;=\; a \wedge b$

(f) $(a \Rightarrow c) \wedge (b \Rightarrow \neg c) \Rightarrow \neg(a \wedge b)$

(g) $a \wedge \neg b \;\Rightarrow\; a \vee b$

(h) $(a \Rightarrow b) \wedge (c \Rightarrow d) \wedge (a \vee c) \Rightarrow (b \vee d)$

(i) $a \wedge \neg a \;\Rightarrow\; b$

(j) $(a \Rightarrow b) \vee (b \Rightarrow a)$

(k)√ $\neg(a \wedge \neg(a \vee b))$

(l) $(\neg a \Rightarrow \neg b) \wedge (a \ne b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$

(m) $(a \Rightarrow \neg a) \Rightarrow \neg a$

(n) $(a \Rightarrow b) \wedge (\neg a \Rightarrow b) \;=\; b$

(o) $(a \Rightarrow b) \Rightarrow a \;=\; a$

(p) $a{=}b \vee a{=}c \vee b{=}c$

(q) $a \wedge b \vee a \wedge \neg b \;=\; a$

(r) $a \Rightarrow (b \Rightarrow a)$

(s) $a \Rightarrow a \wedge b \;=\; a \Rightarrow b \;=\; a \vee b \Rightarrow b$

(t) $(a \Rightarrow a \wedge b) \vee (b \Rightarrow a \wedge b)$

(u) $(a \Rightarrow (p{=}x)) \wedge (\neg a \Rightarrow p) \;=\; p{=}(x \vee \neg a)$

(v) $(a \Rightarrow b \Rightarrow \neg a) \vee (b \wedge c \Rightarrow a \wedge c)$

(w) $a{=}(b \wedge c) \wedge d{=}(\neg b \wedge \neg c) \wedge e{=}((a \vee d){=}c) \;\Rightarrow\; e{=}b$

After trying the question, scroll down to the solution.

§(a)		$a \wedge b$	specialization
	$\Rightarrow$	a	generalization
	$\Rightarrow$	$a \vee b$	

§(b)		$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$	distribute
	$=$	$(a \vee b \vee c) \wedge (a \vee b \vee a) \wedge (a \vee c \vee c) \wedge (a \vee c \vee a) \wedge (b \vee b \vee c) \wedge (b \vee b \vee a) \wedge (b \vee c \vee c) \wedge (b \vee c \vee a)$	
			symmetry and idempotence
	$=$	$\underline{(a \vee b \vee c) \wedge (a \vee b)} \wedge (a \vee c) \wedge (b \vee c)$	absorption
	$=$	$(a \vee b) \wedge (a \vee c) \wedge (b \vee c)$	symmetry
	$=$	$(a \vee b) \wedge (b \vee c) \wedge (c \vee a)$	

§(c)		$\neg a \Rightarrow (a \Rightarrow b)$	portation
	$=$	$\neg a \wedge a \Rightarrow b$	noncontradiction
	$=$	$\bot \Rightarrow b$	base
	$=$	$\top$	

§(d)		$(\underline{a=(b \Rightarrow a)} = a \vee b)$	symmetry of $=$
	$=$	$((b \Rightarrow a)=a = a \vee b)$	associativity of $=$
	$=$	$((b \Rightarrow a) = \underline{a=(a \vee b)})$	symmetry of $=$ and $\vee$
	$=$	$((b \Rightarrow a) = (b \vee a)=a)$	inclusion
	$=$	$\top$	

§(e)		$(\underline{a=(a \Rightarrow b)} = a \wedge b)$	symmetry of $=$
	$=$	$((a \Rightarrow b)=a = a \wedge b)$	associativity of $=$
	$=$	$((a \Rightarrow b) = \underline{a=(a \wedge b)})$	symmetry of $=$
	$=$	$((a \Rightarrow b) = (a \wedge b)=a)$	inclusion
	$=$	$\top$	

§(f)		$(a \Rightarrow c) \wedge (b \Rightarrow \neg c)$	law of conflation
	$\Rightarrow$	$a \wedge b \Rightarrow c \wedge \neg c$	contrapositive law
	$=$	$\neg(c \wedge \neg c) \Rightarrow \neg(a \wedge b)$	antecedent is law of noncontradiction
	$=$	$\top \Rightarrow \neg(a \wedge b)$	identity for $\Rightarrow$
	$=$	$\neg(a \wedge b)$	

§(g)		$a \wedge \neg b$	specialization
	$\Rightarrow$	a	generalization
	$\Rightarrow$	$a \vee b$	

§(h)		$(a \Rightarrow b) \wedge (c \Rightarrow d) \wedge (a \vee c) \Rightarrow (b \vee d)$	portation
	$=$	$(a \Rightarrow b) \wedge (c \Rightarrow d) \Rightarrow (a \vee c \Rightarrow b \vee d)$	conflation
	$=$	$\top$	

§(i)		$a \wedge \neg a \Rightarrow b$	noncontradiction
	$=$	$\bot \Rightarrow b$	base
	$=$	$\top$	

§(j)		$(\underline{a \Rightarrow b}) \vee (\underline{b \Rightarrow a})$	inclusion, twice
	$=$	$\neg a \vee b \vee \neg b \vee a$	symmetry of $\vee$
	$=$	$\underline{a \vee \neg a} \vee \underline{b \vee \neg b}$	excluded middle, twice
	$=$	$\top \vee \top$	idempotence of $\vee$, or base law
	$=$	$\top$	

§(k) see book Subsection 1.0.2

§(l)		$(\neg a \Rightarrow \neg b) \wedge (a \ne b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	law of exclusion
	$=$	$(\neg a \Rightarrow \neg b) \wedge (a = \neg b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	use $a = \neg b$ to replace $\neg b$ with a
	$=$	$(\neg a \Rightarrow a) \wedge (a = \neg b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	indirect proof
	$=$	$a \wedge (a = \neg b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	context to replace second a *by* $\top$, and identity
	$=$	$a \wedge \neg b \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	duality and double negation
	$=$	$\neg(\neg a \vee b) \vee (a \wedge c \Rightarrow b \wedge c)$	inclusion
	$=$	$\neg(a \Rightarrow b) \vee (a \wedge c \Rightarrow b \wedge c)$	context: use left disjunct to simplify the right disjunct; strengthen b to a
	$\Leftarrow$	$\neg(a \Rightarrow b) \vee (a \wedge c \Rightarrow a \wedge c)$	reflexivity and base
	$=$	$\top$	

Here's another solution.

		$(\neg a \Rightarrow \neg b) \wedge (a \ne b) \;\vee\; (a \wedge c \Rightarrow b \wedge c)$	contrapositive, monotonicity
	$\Leftarrow$	$(b \Rightarrow a) \wedge (a \ne b) \;\vee\; (a \Rightarrow b)$	distributivity
	$=$	$((b \Rightarrow a) \vee (a \Rightarrow b)) \wedge ((a \ne b) \vee (a \Rightarrow b))$	material implication twice
	$=$	$(\neg b \vee a \vee \neg a \vee b) \wedge ((a \ne b) \vee (a \Rightarrow b))$	symmetry and associativity
	$=$	$(a \vee \neg a \vee b \vee \neg b) \wedge ((a \ne b) \vee (a \Rightarrow b))$	excluded middle twice, base or idempotent
	$=$	$(a \ne b) \vee (a \Rightarrow b)$	generic unequality
	$=$	$\neg(a = b) \vee (a \Rightarrow b)$	material implication
	$=$	$(a = b) \Rightarrow (a \Rightarrow b)$	antisymmetry (double implication)
	$=$	$(a \Rightarrow b) \wedge (b \Rightarrow a) \Rightarrow (a \Rightarrow b)$	specialization
	$=$	$\top$	

§(m)		$(\underline{a \Rightarrow \neg a}) \Rightarrow \neg a$	material implication
	$=$	$(\underline{\neg a \vee \neg a}) \Rightarrow \neg a$	idempotent
	$=$	$\neg a \Rightarrow \neg a$	reflexive
	$=$	$\top$	

§(n)		$(a \Rightarrow b) \wedge (\neg a \Rightarrow b)$	antidistributive law
	$=$	$a \vee \neg a \;\Rightarrow\; b$	antecedent is law of excluded middle
	$=$	$\top \;\Rightarrow\; b$	identity for $\Rightarrow$
	$=$	b	

Here's another solution.

		$(a \Rightarrow b) \wedge (\neg a \Rightarrow b)$	case analysis
	$=$	**if** a **then** b **else** b **fi**	generic case idempotent
	$=$	b	

§(o)		$(a \Rightarrow b) \Rightarrow a$	*context:* use main consequent to simplify antecedent
	$=$	$\underline{(\bot \Rightarrow b)} \Rightarrow a$	base
	$=$	$\top \Rightarrow a$	identity
	$=$	a	

Here's another solution.

		$(\underline{a \Rightarrow b}) \Rightarrow a$	inclusion
	$=$	$(\neg a \vee b) \Rightarrow a$	inclusion
	$=$	$\underline{\neg(\neg a \vee b)} \vee a$	duality
	$=$	$(\underline{\neg\neg a} \wedge \neg b) \;\vee\; a$	double negation
	$=$	$(a \wedge \neg b) \;\vee\; a$	symmetry of $\vee$
	$=$	$a \;\vee\; (a \wedge \neg b)$	absorption
	$=$	a	

§(p) $a{=}b \lor a{=}c \lor b{=}\underline{c}$ — identity and reflexive laws for $=$

$=\quad a{=}b \lor a{=}c \lor \underline{b{=}((a{=}a){=}c)}$ — symmetry and associative laws for $=$

$=\quad (a{=}b \lor a{=}c) \lor (a{=}b){=}(a{=}c)$ — main $\lor$ distributes over $=$

$=\quad (a{=}b \lor a{=}c \lor a{=}b) = (a{=}b \lor a{=}c \lor a{=}c)$

symmetry, associativity, and idempotence of $\lor$ twice

$=\quad (a{=}b \lor a{=}c) = (a{=}b \lor a{=}c)$ — $=$ is reflexive

$=\quad \top$

Here's another solution.

$a{=}b \lor a{=}c \lor b{=}c$ — double negation

$=\quad \neg\neg(a{=}b) \lor a{=}c \lor b{=}c$ — material implication

$=\quad \neg(a{=}b) \Rightarrow a{=}c \lor b{=}c$ — unequality

$=\quad a{\ne}b \Rightarrow a{=}c \lor b{=}c$ — exclusion

$=\quad a{=}\neg b \Rightarrow a{=}c \lor b{=}c$ — context: use antecedent to modify consequent

$=\quad a{=}\neg b \Rightarrow \neg b{=}c \lor b{=}c$ — exclusion

$=\quad a{=}\neg b \Rightarrow b{\ne}c \lor b{=}c$ — unequality

$=\quad a{=}\neg b \Rightarrow \neg(b{=}c) \lor b{=}c$ — symmetry, excluded middle

$=\quad a{=}\neg b \Rightarrow \top$ — base

$=\quad \top$

§(q) $a{\land}b \lor a{\land}\neg b \;=\; a$ — factor

$=\quad a{\land}(b \lor \neg b) \;=\; a$ — excluded middle

$=\quad a{\land}\top \;=\; a$ — symmetry

$=\quad \top{\land}a \;=\; a$ — identity

$=\quad \top$

§(r) $a{\Rightarrow}(b{\Rightarrow}a)$ — portation

$=\quad a{\land}b \Rightarrow a$ — specialization

$=\quad \top$

§(s) $(a \Rightarrow a{\land}b \;=\; a{\Rightarrow}b \;=\; a{\lor}b \Rightarrow b)$

distribute $\Rightarrow$ over $\land$ in first part; antidistribute $\Rightarrow$ over $\lor$ in last part

$=\quad ((a{\Rightarrow}a) \land (a{\Rightarrow}b) \;=\; a{\Rightarrow}b \;=\; (a{\Rightarrow}b) \land (b{\Rightarrow}b))$

reflexivity of $\Rightarrow$ and identity of $\land$

$=\quad (a{\Rightarrow}b \;=\; a{\Rightarrow}b \;=\; a{\Rightarrow}b)$ — reflexivity of $=$

$=\quad \top$

§(t) $(a \Rightarrow a{\land}b) \lor (b \Rightarrow a{\land}b)$ — anti-distributive

$=\quad a{\land}b \Rightarrow a{\land}b$ — reflexive

$=\quad \top$

§(u) $p{=}(x{\lor}\neg a)$ — case idempotent law

$=\quad$ **if** a **then** $p{=}(x{\lor}\neg a)$ **else** $p{=}(x{\lor}\neg a)$ **fi** — context

$=\quad$ **if** a **then** $p{=}(x{\lor}\neg\top)$ **else** $p{=}(x{\lor}\neg\bot)$ **fi**

$=\quad$ **if** a **then** $p{=}(x{\lor}\bot)$ **else** $p{=}(x{\lor}\top)$ **fi**

$=\quad$ **if** a **then** $p{=}x$ **else** p **fi** — case analysis

$=\quad (a{\Rightarrow}(p{=}x)) \land (\neg a{\Rightarrow}p)$

Here's another solution.

$(a{\Rightarrow}(p{=}x)) \land (\neg a{\Rightarrow}p)$ — case analysis

$=\quad$ **if** a **then** $p{=}x$ **else** p **fi** — identity

$=\quad$ **if** a **then** $p{=}x$ **else** $p{=}\top$ **fi** — case distributive

$=\quad p =$ **if** a **then** x **else** $\top$ **fi** — one-case

$=\quad p = (a{\Rightarrow}x)$ — material implication

$=$	$p = (\neg a \vee x)$	symmetry
$=$	$p = (x \vee \neg a)$	

Here's another solution.

	$(a \Rightarrow (p=x)) \wedge (\neg a \Rightarrow p) \;=\; p=(x \vee \neg a)$	case idempotent
$=$	**if** a **then** $(a \Rightarrow (p=x)) \wedge (\neg a \Rightarrow p) \;=\; p=(x \vee \neg a)$	context
	else $(a \Rightarrow (p=x)) \wedge (\neg a \Rightarrow p) \;=\; p=(x \vee \neg a)$ **fi**	context
$=$	**if** a **then** $(\top \Rightarrow (p=x)) \wedge (\neg \top \Rightarrow p) \;=\; p=(x \vee \neg \top)$	identity, base, identity
	else $(\bot \Rightarrow (p=x)) \wedge (\neg \bot \Rightarrow p) \;=\; p=(x \vee \neg \bot)$ **fi**	base, identity, base
$=$	**if** a **then** $(p=x) \wedge \top \;=\; p=x$	identity, reflexive
	else $\top \wedge p \;=\; p=\top$ **fi**	identity, reflexive
$=$	**if** a **then** $\top$ **else** $\top$ **fi**	case idempotent
$=$	$\top$	

§(v)

	$(a \Rightarrow b \Rightarrow \neg a) \vee (b \wedge c \Rightarrow a \wedge c)$	continuing implication
$=$	$(a \Rightarrow b) \wedge (b \Rightarrow \neg a) \;\vee\; (b \wedge c \Rightarrow a \wedge c)$	contrapositive and double negation
$=$	$(a \Rightarrow b) \wedge (a \Rightarrow \neg b) \;\vee\; (b \wedge c \Rightarrow a \wedge c)$	distributive
$=$	$(a \Rightarrow b \wedge \neg b) \;\vee\; (b \wedge c \Rightarrow a \wedge c)$	noncontradiction
$=$	$(a \Rightarrow \bot) \;\vee\; (b \wedge c \Rightarrow a \wedge c)$	double negation and indirect proof
$=$	$\neg a \;\vee\; (b \wedge c \Rightarrow a \wedge c)$	context
$=$	$\neg a \;\vee\; (b \wedge c \Rightarrow \top \wedge c)$	identity
$=$	$\neg a \;\vee\; (b \wedge c \Rightarrow c)$	specialization
$=$	$\neg a \;\vee\; \top$	base
$=$	$\top$	

Here's another solution.

	$(a \Rightarrow b \Rightarrow \neg a) \vee (b \wedge c \Rightarrow a \wedge c)$	continuing implication
$=$	$(a \Rightarrow b) \wedge (b \Rightarrow \neg a) \;\vee\; (b \wedge c \Rightarrow a \wedge c)$	monotonic
$\Leftarrow$	$(a \Rightarrow b) \wedge (a \Rightarrow \neg b) \;\vee\; (b \Rightarrow a)$	distributive
$=$	$((a \Rightarrow b) \vee (b \Rightarrow a)) \;\wedge\; ((a \Rightarrow \neg b) \vee (b \Rightarrow a))$	inclusion, twice
$=$	$(\neg a \vee b \vee \neg b \vee a) \;\wedge\; ((a \Rightarrow \neg b) \vee (b \Rightarrow a))$	symmetry of $\vee$
$=$	$(a \vee \neg a \vee b \vee \neg b) \;\wedge\; ((a \Rightarrow \neg b) \vee (b \Rightarrow a))$	excluded middle, twice
$=$	$(\top \vee \top) \;\wedge\; ((a \Rightarrow \neg b) \vee (b \Rightarrow a))$	idempotence of $\vee$, or base law
$=$	$\top \;\wedge\; ((a \Rightarrow \neg b) \vee (b \Rightarrow a))$	identity
$=$	$(a \Rightarrow \neg b) \vee (b \Rightarrow a)$	contrapositive
$=$	$(a \Rightarrow \neg b) \vee (\neg a \Rightarrow \neg b)$	antidistributive
$=$	$a \wedge \neg a \Rightarrow \neg b$	noncontradiction
$=$	$\bot \Rightarrow \neg b$	base
$=$	$\top$	

§(w)

	$a=(b \wedge c) \wedge d=(\neg b \wedge \neg c) \wedge e=((a \vee d)=c)$	
		context: use first two conjuncts to modify last conjunct
$=$	$a=(b \wedge c) \wedge d=(\neg b \wedge \neg c) \wedge e=((b \wedge c \;\vee\; \neg b \wedge \neg c)=c)$	specialization
$\Rightarrow$	$e=((b \wedge c \;\vee\; \neg b \wedge \neg c)=c)$	equality and difference
$=$	$e=((b=c)=c)$	associativity
$=$	$(e=b)=(c=c)$	reflexivity, identity
$=$	$e=b$	