53 (von Neumann numbers)

(a) Is there any harm in adding the axioms

$0 = \{null\}$ the empty set

$n+1 = \{n, \sim n\}$ for each natural $n$

(b) What correspondence is induced by these axioms between the arithmetic operations and the set operations?

(c) Is there any harm in adding the axioms

$0 = \{null\}$ the empty set

$i+1 = \{i, \sim i\}$ for each integer $i$

After trying the question, scroll down to the solution.

(a) Is there any harm in adding the axioms

$0 \;=\; \{null\}$ the empty set

$n{+}1 \;=\; \{n, \sim n\}$ for each natural $n$

§ John von Neumann used these axioms as part of his demonstration that set theory could be used to construct all of mathematics. In the second axiom, in place of $n$, if we substitute $0$, then $1$, then $2$, and so on, we find

$$1 \;=\; \{0, \sim 0\} \;=\; \{0, \sim\{null\}\} \;=\; \{0, null\} \;=\; \{0\}$$
$$2 \;=\; \{1, \sim 1\} \;=\; \{1, \sim\{0\}\} \;=\; \{1, 0\}$$
$$3 \;=\; \{2, \sim 2\} \;=\; \{2, \sim\{1, 0\}\} \;=\; \{2, 1, 0\}$$

And in general,

$$n \;=\; \{0,..n\}$$

Each natural is equated to the set of all smaller naturals. Note $\$n = n$. The only harm would be if these axioms were inconsistent with the axioms we already have. The only axiom we have that directly relates a number and a set is $\{A\} \mp A$ so with the new axioms we now have $\{n\} \mp \{0,..n\}$, or $n \mp 0,..n$, but that's all right; so no harm there. In string and list theory, we can index with a natural number, and we can also index with a set, yielding a set as result. For example,

$$[0;\, 0]\; 1 \;=\; 0$$
$$[0;\, 0]\; \{0\} \;=\; \{0\}$$

Now if $1 = \{0\}$ then $0 = \{0\}$, and so $0{=}1$. That's harm. To include von Neumann's axioms, we would have to withdraw the ability of sets to be used as indexes.

(b) What correspondence is induced by these axioms between the arithmetic operations and the set operations?

§ I'll start with an easy one: $n{\le}m$. I could say $n{\le}m \;=\; \$n{\le}\$m$, but since the result of $\$$ is a number, that doesn't make the desired correspondence. So I'll say

$$n{\le}m \;=\; n{\subseteq}m$$

Two more easy and related operations are

$$n{\uparrow}m \;=\; n{\cup}m$$
$$n{\downarrow}m \;=\; n{\cap}m$$

To add $1$ we are given

$$n{+}1 \;=\; \{n, \sim n\}$$

or

$$n{+}1 \;=\; n{\cup}\{n\}$$

Now we can define addition inductively:

$$n{+}0 \;=\; n$$
$$n{+}(m{+}1) \;=\; (n{+}m){+}1 \;=\; (n{+}m){\cup}\{n{+}m\}$$

Multiplication can be defined in terms of addition, inductively, as follows.

$$n{\times}0 = 0$$
$$n{\times}(m{+}1) \;=\; n{\times}m + n$$

Since we already have addition in terms of set operations, now we have multiplication, but it is not very intuitive. To be intuitive, I could write

$$n{\times}m \;=\; \{0,..n{\times}m\}$$

but that doesn't tell us how to multiply.

(c) Is there any harm in adding the axioms

$0 \;=\; \{null\}$ the empty set

$i{+}1 \;=\; \{i, \sim i\}$ for each integer $i$

§ These two axioms don't contradict any that we already have, but they contradict each other when $i$ is $-1$. Instantiate the second axiom:

| | | |
|---|---|---|
| | $-1 + 1\ =\ \{-1, \sim -1\}$ | arithmetic |
| $=$ | $0 = \{-1, \sim -1\}$ | use first axiom |
| $=$ | $\{null\} = \{-1, \sim -1\}$ | equation axiom of set theory |
| $=$ | $null\ =\ -1, \sim -1$ | |
| $\Longrightarrow$ | $-1\text{: } null$ | conjoin an instance of an axiom for $null$ |
| $=$ | $-1\text{: } null\text{: } 0$ | transitivity |
| $\Longrightarrow$ | $-1\text{: } 0$ | $-1$ and $0$ are elements |
| $=$ | $-1 = 0$ | |
| $=$ | $\bot$ | |

The first line is an instance of an axiom, so it's a theorem. The last line is an antitheorem, so we have an inconsistency.

But wait a minute. In Exercise 46 we looked at sets that can have a negative number of elements. In part (a) of this exercise we see

$$n\ =\ \{0,..n\}$$

and so $n$ is a set that contains $n$ elements $\$n = n$ . So maybe $i$ can contain $i$ elements, even if $i$ is negative. In the calculation above, we arrive at the line

$$null\ =\ -1, \sim -1$$

in which $-1$ is one element, and $\sim -1$ is $-1$ elements, and so the union $-1, \sim -1$ is $0$ elements. Some of the axioms of bunch theory need to be weakened or withdrawn to make room for bunches and sets with a negative number of elements. And then we can have the axioms proposed in Exercise 53(c).