Let \( \otimes \) be a two-operand infix operator (precedence 3) with natural operands and an extended natural result. Informally, \( n \otimes m \) means “the number of times that \( n \) is a factor of \( m \)”. It is defined by the following two axioms.

\[
\begin{align*}
m : n \times \text{nat} & \lor n \otimes m = 0 \\
n \neq 0 & \Rightarrow n \otimes (m \times n) = n \otimes m + 1
\end{align*}
\]

(a) Make a 3×3 chart of the values of \( (0,3) \otimes (0,3) \).

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 0 & 0 \\
1 & \infty & \infty & \infty \\
2 & \infty & 0 & 1 \\
\end{array}
\]

(b) Show that the axioms become inconsistent if the antecedent of the second axiom is removed.

\[
\begin{align*}
0 \otimes 0 &= 0 \otimes (1 \times 0) = 0 \otimes 1 + 1 = 0 + 1 = 1 \\
0 \otimes 0 &= 0 \otimes (0 \times 0) = 0 \otimes 0 + 1
\end{align*}
\]

Hence \( 1 = 1 + 1 \).

(c) How should we change the axioms to allow \( \otimes \) to have extended natural operands?

\[
\begin{align*}
\text{From the first axiom, instantiating with } m=\infty \text{ and } n=1, \text{ we get} \\
\infty : 1 \times \text{nat} & \lor 1 \otimes \infty = 0 \\
\equiv & \quad \perp \lor 1 \otimes \infty = 0 \\
\equiv & \quad 1 \otimes \infty = 0
\end{align*}
\]

From the second axiom, instantiating with \( m=\infty \) and \( n=1 \), we get

\[
\begin{align*}
1 \neq 0 & \Rightarrow 1 \otimes (\infty \times 1) = 1 \otimes \infty + 1 \\
\equiv & \quad 1 \otimes \infty = 1 \otimes \infty + 1 \\
\equiv & \quad 0 = 0 + 1
\end{align*}
\]

So we can’t leave the axioms as they are. We can change \( \text{nat} \) to \( x\text{nat} \) in the first axiom; now for \( n \neq 0 \) we have \( n \otimes \infty = \infty \). Perhaps we don’t want \( \infty \otimes \infty = \infty \), so perhaps we should weaken the second axioms to \( 0 < n < \infty \Rightarrow n \otimes (m \times n) = n \otimes m + 1 \). We now have no answer for \( \infty \otimes m \), and I don’t know what it should be.