Let $\otimes$ be a two-operand infix operator (precedence 3) with natural operands and an extended natural result. Informally, $n \otimes m$ means “the number of times that $n$ is a factor of $m$”. It is defined by the following two axioms.

- $m: n \times \text{nat} \lor n \otimes m = 0$
- $n \neq 0 \Rightarrow n \otimes (m \times n) = n \otimes m + 1$

(a) Make a $3 \times 3$ chart of the values of $(0,..3) \otimes (0,..3)$.
(b) Show that the axioms become inconsistent if the antecedent of the second axiom is removed.
(c) How should we change the axioms to allow $\otimes$ to have extended natural operands?

After trying the question, scroll down to the solution.
(a) Make a $3 \times 3$ chart of the values of $(0,..3) \otimes (0,..3)$.

\[
\begin{array}{c|ccc}
0 & 0 & 0 \\
1 & \infty & \infty & \infty \\
2 & \infty & 0 & 1 \\
\end{array}
\]

(b) Show that the axioms become inconsistent if the antecedent of the second axiom is removed.

\[
0 \otimes 0 = 0 \otimes (1 \times 0) = 0 \otimes 1 + 1 = 0 + 1 = 1 \\
0 \otimes 0 = 0 \otimes (0 \times 0) = 0 \otimes 0 + 1
\]

Hence $1 = 1 + 1$.

(c) How should we change the axioms to allow $\otimes$ to have extended natural operands?

From the first axiom, instantiating with $m=\infty$ and $n=1$, we get

\[
\infty \times 1 \times \text{nat} \lor 1 \otimes \infty = 0 \\
\equiv \quad \bot \lor 1 \otimes \infty = 0 \\
\equiv \quad 1 \otimes \infty = 0
\]

From the second axiom, instantiating with $m=\infty$ and $n=1$, we get

\[
1 + 0 \Rightarrow 1 \otimes (\infty \times 1) = 1 \otimes \infty + 1 \\
\equiv \quad 1 \otimes \infty = 1 \otimes \infty + 1
\]

now use what we got from the first axiom

\[
\equiv \quad 0 = 0 + 1
\]

So we can't leave the axioms as they are. We can change $\text{nat}$ to $\text{xnat}$ in the first axiom; now for $n + 0$ we have $n \otimes \infty = \infty$. Perhaps we don't want $\infty \otimes \infty = \infty$, so perhaps we should weaken the second axioms to $0 < n < \infty \Rightarrow n \otimes (m \times n) = n \otimes m + 1$. We now have no answer for $\infty \otimes m$, and I don't know what it should be.