A hyperbunch is like a bunch except that each element can occur a number of times other than just zero times (absent) or one time (present). The order of elements remains insignificant. (A hyperbunch does not have a characteristic predicate, but a characteristic function with numeric result.) Design notations and axioms for each of the following kinds of hyperbunch.

(a) multibunch: an element can occur any natural number of times. For example, a multibunch can consist of one 2, two 7s, three 5s, and zero of everything else. (Note: a packaged multibunch is called either a multiset or a bag.)

Any bunch is also a multibunch. Let \( A \) and \( B \) be multibunches. Let \( x \) and \( y \) be elements (number, character, binary, set). Let \( n \) be a natural. Then \( A \oplus B \) and \( n \otimes A \) are multibunches, \( x \# A \) is natural, and \( A = B \) and \( A : B \) are binary, with axioms

\[
\begin{align*}
(x \# y) &= 1 \\
x \# y &= (x \# y) = 0 \\
(n \otimes x) &= x = n = 1 \\
x \# (A \oplus B) &= \max 0 ((x \# A) - (x \# B)) \\
x \# (A \odot B) &= \max 0 ((x \# A) - (x \# B)) \\
x \# (n \otimes A) &= n \times (x \# A) \\
A \oplus \text{null} &= A \\
A \odot \text{null} &= A \\
\text{null} \odot A &= \text{null} \\
A = B &\Rightarrow (x \# A) = (x \# B) \\
A : B &\Rightarrow (x \# A) \leq (x \# B) \\
1 \otimes A &= A
\end{align*}
\]

It is not obvious whether and how I should let ordinary element operators distribute over multibunch “union” \( \oplus \) so I have left it out.

(b) wholebunch: an element can occur any integer number of times.

This is like multibunches, except of course that \( n \) can be any integer. I remove the axioms

\[
\begin{align*}
(x \# A \odot B) &= \max 0 ((x \# A) - (x \# B)) \\
\text{null} \odot A &= \text{null}
\end{align*}
\]

and add the axioms

\[
\begin{align*}
x \# (A \odot B) &= (x \# A) - (x \# B) \\
A \odot (B \odot C) &= (A \odot B) \odot C
\end{align*}
\]

(c) fuzzybunch: an element can occur any real number of times from 0 to 1 inclusive.

I use the same expressions again except that \( A \oplus B \) and \( A \odot B \) are replaced by \( A \otimes B \) and \( A \odot B \), and \( n \) is real and \( 0 \leq n \leq 1 \). The axioms are

\[
\begin{align*}
(x \# x) &= 1 \\
x \# y &= (x \# y) = 0 \\
(n \otimes x) &= x = n = 1 \\
x \# (A \otimes B) &= \max 0 ((x \# A) - (x \# B)) \\
x \# (A \odot B) &= \min (x \# A) (x \# B) \\
x \# (n \otimes A) &= n \times (x \# A) \\
A \otimes \text{null} &= A \\
A \odot \text{null} &= \text{null} \\
0 \otimes A &= \text{null} \\
1 \otimes A &= A \\
A = B &\Rightarrow (x \# A) = (x \# B) \\
A : B &\Rightarrow (x \# A) \leq (x \# B)
\end{align*}
\]