Given natural number $n$, a theory maintains a subbunch of $0...n$. The operations are: `mkempty`, which makes the bunch empty; `insert x`, which inserts $x$ into the bunch; `remove x`, which removes $x$ if it was there, and `check x` which tells whether $x$ is there by assigning to a user's binary variable $u$.

(a) Design axioms that are weak enough to allow other operations to be added to the theory.
(b) Implement your theory of part (a) as a list of binary values.
(c) Transform your implementation of part (b) to one that maintains a list of natural numbers.

After trying the question, scroll down to the solution.
(a) Design axioms that are weak enough to allow other operations to be added to the theory.

§ The axioms use an auxiliary operation \(\text{preserve } x\) that does not affect whether \(x\) is in the bunch.

- \(\text{mkempty. preserve } x. \text{check } x \Rightarrow \neg u'\)
- \(\text{insert } x. \text{preserve } x. \text{check } x \Rightarrow u'\)
- \(\text{remove } x. \text{preserve } x. \text{check } x \Rightarrow \neg u'\)
- \(\text{preserve } x \Leftarrow \text{ok}\)
- \(\text{preserve } x \Leftarrow \text{check } x\)
- \(\text{preserve } x \Leftarrow \text{insert } y \land x \neq y\)
- \(\text{preserve } x \Leftarrow \text{remove } y \land x \neq y\)
- \(\text{preserve } x \Leftarrow \text{preserve } x. \text{preserve } x\)

(b) Implement your theory of part (a) as a list of binary values.

(c) Transform your implementation of part (b) to one that maintains a list of natural numbers.