46 (hyperbunch) A hyperbunch is like a bunch except that each element can occur a number of times other than just zero times (absent) or one time (present). The order of elements remains insignificant. (A hyperbunch does not have a characteristic predicate, but a characteristic function with numeric result.) Design notations and axioms for each of the following kinds of hyperbunch.

(a) multibunch: an element can occur any natural number of times. For example, a multibunch can consist of one 2, two 7s, three 5s, and zero of everything else. (Note: the equivalent for sets is called either a multiset or a bag.)

(b) wholebunch: an element can occur any integer number of times.

(c) fuzzybunch: an element can occur any real number of times from 0 to 1 inclusive.

After trying the question, scroll down to the solution.

(a) multibunch: an element can occur any natural number of times. For example, a multibunch can consist of one 2, two 7s, three 5s, and zero of everything else. (Note: a packaged multibunch is called either a multiset or a bag.)

§ Any bunch is also a multibunch. Let A and B be multibunches. Let x and y be elements (number, character, binary, set). Let n be a natural. Then

$A \oplus B$	A union B
$A \ominus B$	A take away B
$n \otimes A$	n times A

are multibunches,

$x\#A$	the number of occurrences of x in A

is natural, and

$A=B$	A equals B
$A:B$	A is included in B

are binary, with axioms

$(x\#x) = 1$
$x \neq y \;=\; (x\#y) = 0$
$(n \otimes x) = x \;=\; n=1$
$x\#(A \oplus B) \;=\; (x\#A) + (x\#B)$
$x\#(A \ominus B) \;=\; 0 \uparrow ((x\#A) - (x\#B))$
$x\#(n \otimes A) \;=\; n \times (x\#A)$
$A \oplus null \;=\; A$
$A \ominus null \;=\; A$
$null \ominus A \;=\; null$
$0 \otimes A \;=\; null$
$1 \otimes A \;=\; A$

$A \oplus B \;=\; B \oplus A$
$A \oplus (B \oplus C) \;=\; (A \oplus B) \oplus C$
$A \ominus (B \oplus C) \;=\; (A \ominus B) \ominus C$
$n \otimes (m \otimes A) \;=\; (n \times m) \otimes A$
$n \otimes (A \oplus B) \;=\; (n \otimes A) \oplus (n \otimes B)$
$(n \otimes A) \oplus (m \otimes A) \;=\; (n+m) \otimes A$
$A=B \;\Rightarrow\; (x\#A)=(x\#B)$
$A:B \;\Rightarrow\; (x\#A) \le (x\#B)$

It is not obvious whether and how I should let ordinary element operators distribute over multibunch "union" $\oplus$ so I have left it out.

(b) wholebunch: an element can occur any integer number of times.

§ This is like multibunches, except of course that n can be any integer. I remove the axioms

$x\#(A \ominus B) \;=\; 0 \uparrow ((x\#A) - (x\#B))$
$null \ominus A \;=\; null$

and add the axioms

$x\#(A \ominus B) \;=\; (x\#A) - (x\#B)$
$A \ominus (B \ominus C) \;=\; (A \ominus B) \oplus C$

(c) fuzzybunch: an element can occur any real number of times from 0 to 1 inclusive.

§ I use the same expressions again except that $A \oplus B$ and $A \ominus B$ are replaced by $A \ovee B$ and $A \owedge B$, and n is real and $0 \le n \le 1$. The axioms are

$(x\#x) = 1$
$x \neq y \;=\; (x\#y) = 0$
$(n \otimes x) = x \;=\; n=1$
$x\#(A \ovee B) \;=\; (x\#A) \uparrow (x\#B)$
$x\#(A \owedge B) \;=\; (x\#A) \downarrow (x\#B)$
$x\#(n \otimes A) \;=\; n \times (x\#A)$
$A \ovee null \;=\; A$
$A \owedge null \;=\; null$
$0 \otimes A \;=\; null$
$1 \otimes A \;=\; A$

$A \ovee B \;=\; B \ovee A$
$A \owedge B \;=\; B \owedge A$
$A \ovee (B \ovee C) \;=\; (A \ovee B) \ovee C$
$A \owedge (B \owedge C) \;=\; (A \owedge B) \owedge C$
$A \owedge (B \ovee C) \;=\; (A \owedge B) \ovee (A \owedge C)$
$A \ovee (B \owedge C) \;=\; (A \ovee B) \owedge (A \ovee C)$
$n \otimes (m \otimes A) \;=\; (n \times m) \otimes A$
$n \otimes (A \ovee B) \;=\; (n \otimes A) \ovee (n \otimes B)$
$n \otimes (A \owedge B) \;=\; (n \otimes A) \owedge (n \otimes B)$
$A=B \;\Rightarrow\; (x\#A)=(x\#B)$
$A:B \;\Rightarrow\; (x\#A) \le (x\#B)$