(a) multibunch: an element can occur any natural number of times. For example, a multibunch can consist of one 2, two 7s, three 5s, and zero of everything else. (Note: a packaged multibunch is called either a multiset or a bag.)

Any bunch is also a multibunch. Let $A$ and $B$ be multibunches. Let $x$ and $y$ be elements (number, character, binary, set). Let $n$ be a natural. Then

$$A \oplus B \quad \text{A union } B$$

$$A \ominus B \quad \text{A take away } B$$

$$n \otimes A \quad n \text{ times } A$$

are multibunches,

$$x \# A \quad \text{the number of occurrences of } x \text{ in } A$$

is natural, and

$$A = B \quad A \text{ equals } B$$

$$A : B \quad A \text{ is included in } B$$

are binary, with axioms

\[
\begin{align*}
(x \# y) &= 1 & A \oplus B &= B \oplus A \\
x + y &= (x \# y) = 0 & A \oplus (B \oplus C) &= (A \oplus B) \oplus C \\
(n \otimes x) &= x = n \cdot 1 & A \ominus (B \ominus C) &= (A \ominus B) \ominus C \\
x \# (A \oplus B) &= (x \# A) + (x \# B) & n \varotimes (m \otimes A) &= (n \times m) \otimes A \\
x \# (A \ominus B) &= \max 0 ((x \# A) - (x \# B)) & n \varominus (A \ominus B) &= (n \ominus A) \ominus (n \ominus B) \\
x \# n \otimes A &= n \times (x \# A) & (n \otimes A) \oplus (m \otimes A) &= (n + m) \otimes A \\
A \oplus \text{null} &= A \\
A \ominus \text{null} &= A \\
\text{null} \ominus A &= \text{null} & A = B \implies (x \# A) = (x \# B) \\
0 \otimes A &= \text{null} & A : B \implies (x \# A) \leq (x \# B) \\
1 \otimes A &= A
\end{align*}
\]

It is not obvious whether and how I should let ordinary element operators distribute over multibunch “union” $\oplus$ so I have left it out.

(b) wholebunch: an element can occur any integer number of times.

This is like multibunches, except of course that $n$ can be any integer. I remove the axioms

$$x \# (A \ominus B) = \max 0 ((x \# A) - (x \# B))$$

$$\text{null} \ominus A = \text{null}$$

and add the axioms

$$x \# (A \ominus B) = (x \# A) - (x \# B)$$

$$A \ominus (B \ominus C) = (A \ominus B) \ominus C$$

(c) fuzzybunch: an element can occur any real number of times from 0 to 1 inclusive.

I use the same expressions again except that $A \oplus B$ and $A \ominus B$ are replaced by $A \oplus B$ and $A \ominus B$, and $n$ is real and $0 \leq n \leq 1$. The axioms are

\[
\begin{align*}
(x \# x) &= 1 & A \oplus B &= B \oplus A \\
x + y &= (x \# y) = 0 & A \oplus (B \oplus C) &= (A \oplus B) \oplus C \\
(n \otimes x) &= x = n \cdot 1 & A \ominus (B \ominus C) &= (A \ominus B) \ominus C \\
x \# (A \oplus B) &= \max (x \# A) (x \# B) & A \ominus (B \ominus C) &= (A \ominus B) \ominus (A \ominus C) \\
x \# (A \ominus B) &= \min (x \# A) (x \# B) & A \ominus (B \ominus C) &= (A \ominus B) \ominus (A \ominus C) \\
x \# n \otimes A &= n \times (x \# A) & A \ominus (B \ominus C) &= (A \ominus B) \ominus (A \ominus C)
\end{align*}
\]
<table>
<thead>
<tr>
<th>Rule</th>
<th>Equivalent Expression</th>
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<tbody>
<tr>
<td>( A \otimes \text{null} = A )</td>
<td>( n \otimes (m \otimes A) = (n \times m) \otimes A )</td>
</tr>
<tr>
<td>( A \otimes \text{null} = \text{null} )</td>
<td>( n \otimes (A \otimes B) = (n \otimes A) \otimes (n \otimes B) )</td>
</tr>
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</tr>
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<td>( 1 \otimes A = A )</td>
<td>( A = B \Rightarrow (x # A) = (x # B) )</td>
</tr>
<tr>
<td></td>
<td>( A : B \Rightarrow (x # A) \leq (x # B) )</td>
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