In a graphical program, a pixel might be identified by its Cartesian co-ordinates \(x\) and \(y\), or by its polar co-ordinates \(r\) (radius, or distance from the origin) and \(a\) (angle in radians counter-clockwise from the \(x\) axis). An operation written using one kind of co-ordinates may need to be transformed into the other kind of co-ordinates.

(a) What is the data transformer to transform from Cartesian to polar co-ordinates?

(b) In Cartesian co-ordinates, one of the operations on a pixel is \textit{translate}, which moves a pixel from position \(x\) and \(y\) to position \(x+u\) and \(y+v\).

\[
\text{translate} \equiv x := x + u, \quad y := y + v
\]

Use the data transformer from (a) to transform operation \textit{translate} from Cartesian to polar co-ordinates.

(c) What is the data transformer to transform from polar to Cartesian co-ordinates?

(d) In polar co-ordinates, one of the operations on a pixel is \textit{rotate} by \(d\) radians.

\[
\text{rotate} \equiv a := a + d
\]

Use the data transformer from (c) to transform operation \textit{rotate}.

After trying the question, scroll down to the solution.
§ In polar co-ordinates, one of the operations on a pixel is \( \text{rotate} \), which moves a pixel from position \( x \) and \( y \) to position \( x+u \) and \( y+v \).

\( \text{translate} = x:=x+u \land y:=y+v \)

Use the data transformer from (a) to transform operation \( \text{translate} \) from Cartesian to polar co-ordinates.

\[ \forall x, \, y \cdot x = r \times \cos a \land y = r \times \sin a \]
\[ \Rightarrow \exists x', \, y' \cdot x' = r' \times \cos a' \land y' = r' \times \sin a' \land \text{translate} \]
\[ = \forall x, \, y \cdot x = r \times \cos a \land y = r \times \sin a \]
\[ \Rightarrow \exists x', \, y' \cdot x' = r' \times \cos a' \land y' = r' \times \sin a' \land (x:=x+u \land y:=y+v) \]
\[ = \forall x, \, y \cdot x = r \times \cos a \land y = r \times \sin a \]
\[ \Rightarrow r' \times \cos a' = (r \times \cos a) + u \land r' \times \sin a' = (r \times \sin a) + v \]

This is not yet a program. It appears that the way to get a \( \text{translate} \) program in polar co-ordinates is to transform to Cartesian, \( \text{translate} \) in Cartesian, then transform back to polar.

\[ = \forall x, \, y \cdot \exists y' : r' := (x \times y + 2 y)^2 + u \land a := \arctan (y/x) \]

(c) What is the data transformer to transform from polar to Cartesian co-ordinates?

The same transformer from part (a) works, but for this direction, it is more conveniently rewritten, using trigonometric identities, as

\[ r = (x^2 + y^2)^{1/2} \land a = \arctan (y/x) \]

to use one-point laws to get rid of quantifications \( \forall r, \, a' \) and \( \exists r', \, a' \).

(d) In polar co-ordinates, one of the operations on a pixel is \( \text{rotate} \) by \( d \) radians.

\( \text{rotate} = a := a + d \)

Use the data transformer from (c) to transform operation \( \text{rotate} \).

\[ \forall r, \, a' \cdot r = (x^2 + y^2)^{1/2} \land a = \arctan (y/x) \]
\[ \Rightarrow \exists r', \, a' : r' = (x^2 + y^2)^{1/2} \land a' = \arctan (y'/x') \land \text{rotate} \]
\[ = \forall r, \, a' \cdot r = (x^2 + y^2)^{1/2} \land a = \arctan (y/x) \]
\[ \Rightarrow \exists r', \, a' : r' = (x^2 + y^2)^{1/2} \land a' = \arctan (y'/x') \land (a := a + d) \]
\[ = \forall r, \, a' \cdot r = (x^2 + y^2)^{1/2} \land a = \arctan (y/x) \]
\[ \Rightarrow (x^2 + y^2)^{1/2} = r \land \arctan (y'/x') = a + d \]
\[ = (x^2 + y^2)^{1/2} = (x^2 + y^2)^{1/2} \land \arctan (y'/x') = \arctan (y/x) + d \]
\[ = x^2 + y^2 = x^2 + y^2 \land \arctan (y'/x') = \arctan (y/x) + d \]

This is not yet a program. It appears that the way to get a \( \text{rotate} \) program in Cartesian co-ordinates is to transform to polar, \( \text{rotate} \) in polar, then transform back to Cartesian.
\begin{align*}
\texttt{var } & r, a: \text{ real} \quad r := (x^2 + y^2)^{1/2}. \quad a := \arctan (y/x) + d. \\
x := r \times \cos a. \quad y := r \times \sin a
\end{align*}