A theory provides three names: *set*, *flip*, and *ask*. It is presented by an implementation. Let *u*: *bin* be the user's variable, and let *v*: *bin* be the implementer's variable. The axioms are

\[ set \equiv v := \top \]
\[ flip \equiv v := \neg v \]
\[ ask \equiv u := v \]

(a) √ Replace *v* with *w*: *nat* according to the data transformer *v* = *even* *w*.

(b) Replace *v* with *w*: *nat* according to the data transformer \((w=0 \Rightarrow v) \land (w=1 \Rightarrow \neg v)\). Is anything wrong?

(c) Replace *v* with *w*: *nat* according to \((v \Rightarrow w=0) \land (\neg v \Rightarrow w=1)\). Is anything wrong?

After trying the question, scroll down to the solution.
(a)√ Replace \( v \) with \( w: \text{nat} \) according to the data transformer \( v = \text{even } w \).

§ see book Section 7.2

(b) Replace \( v \) with \( w: \text{nat} \) according to the data transformer \( (w=0 \Rightarrow v) \land (w=1 \Rightarrow \neg v) \). Is anything wrong?

§ Operation \textit{set} becomes

\[
\forall v \cdot (w=0 \Rightarrow v) \land (w=1 \Rightarrow \neg v) \Rightarrow \exists v' \cdot (w'=0 \Rightarrow v') \land (w'=1 \Rightarrow \neg v') \land (v' = \top)
\]

\[
= u'=u \land w'+1
\]

Operation \textit{flip} becomes

\[
\forall v \cdot (w=0 \Rightarrow v) \land (w=1 \Rightarrow \neg v) \Rightarrow \exists v' \cdot (w'=0 \Rightarrow v') \land (w'=1 \Rightarrow \neg v') \land (v' = \neg v)
\]

\[
= u'=u \land (w \neq 0 \Rightarrow w'+1) \land (w+1 \Rightarrow w'+0)
\]

Operation \textit{ask} becomes

\[
\forall v \cdot (w=0 \Rightarrow v) \land (w=1 \Rightarrow \neg v) \Rightarrow \exists v' \cdot (w'=0 \Rightarrow v') \land (w'=1 \Rightarrow \neg v') \land (u' = v)
\]

\[
= (w \neq 0 \Rightarrow w'+0 \land \neg u') \land (w+1 \Rightarrow w'+1 \land u')
\]

\[
= (w=0 \land w'+1 \land u') \lor (w=1 \land w'+0 \land \neg u')
\]

Something is wrong. Although \( (w=0 \Rightarrow v) \land (w=1 \Rightarrow \neg v) \) is a data transformer, it is a rather weak one because when \( w' \) is neither 0 nor 1, it doesn't constrain \( v \). So the result is that \textit{ask} is transformed into something that's unimplementable.

(c) Replace \( v \) with \( w: \text{nat} \) according to \( (v \Rightarrow w=0) \land (\neg v \Rightarrow w=1) \). Is anything wrong?

§ Operation \textit{set} becomes

\[
\forall v \cdot (v \Rightarrow w=0) \land (\neg v \Rightarrow w=1) \Rightarrow \exists v' \cdot (v' \Rightarrow w'=0) \land (\neg v' \Rightarrow w'=1) \land (v' = \top)
\]

\[
= w: 0,1 \Rightarrow (w'=0)
\]

Operation \textit{flip} becomes

\[
\forall v \cdot (v \Rightarrow w=0) \land (\neg v \Rightarrow w=1) \Rightarrow \exists v' \cdot (v' \Rightarrow w'=0) \land (\neg v' \Rightarrow w'=1) \land (v' = \neg v)
\]

\[
= w: 0,1 \Rightarrow (w'=1-w)
\]

Operation \textit{ask} becomes

\[
\forall v \cdot (v \Rightarrow w=0) \land (\neg v \Rightarrow w=1) \Rightarrow \exists v' \cdot (v' \Rightarrow w'=0) \land (\neg v' \Rightarrow w'=1) \land (u' = v)
\]

\[
= w: 0,1 \Rightarrow (u'=w=0)
\]

Something is wrong. We have been transforming with something that isn't a transformer; it's too strong.

\[
\forall w \cdot \exists v \cdot (v \Rightarrow w=0) \land (\neg v \Rightarrow w=1)
\]

\[
= \forall w \cdot w=0 \lor w=1
\]

\[
= \bot
\]

The last line isn't a theorem, so neither is the first. Nothing constrains the implementation to start in a state where \( w=0 \lor w=1 \). If it starts with \( w=2 \), then \textit{set} might not set \( w \) to 0, after which \textit{ask} will give the wrong answer.