436 (leafy tree) A leafy tree is a tree with information residing only at the leaves. Design appropriate axioms for a binary leafy data-tree.

After trying the question, scroll down to the solution.

§ The following axioms constitute a strong theory of leafy trees.

    leaf: $X \to tree$
    graft: $tree \to tree \to tree$
    leaf X, *graft* B B: B $\Rightarrow$ *tree*: B
    graft t $u \neq$ *leaf* x
    leaf $x =$ *leaf* y $=$ $x=y$
    graft t $u =$ *graft* v w $=$ $t=v \wedge u=w$
    left (*graft* t u) $=$ t
    right (*graft* t u) $=$ u
    data (*leaf* x) $=$ x

I have used a function *leaf* to convert a data item to a one-item tree, and another function *data* to retrieve it again. Another, simpler, approach is to consider that a data item is already a one-item tree. In that case, *leaf* and *data* aren't needed. The axioms are:

    X: *tree*

    graft: $tree \to tree \to tree$
    X, *graft* B B: B $\Rightarrow$ *tree*: B
    $\neg$ *graft* t u: X
    graft t $u =$ *graft* v w $=$ $t=v \wedge u=w$
    left (*graft* t u) $=$ t
    right (*graft* t u) $=$ u