420 Suppose we define **while** b **do** P **od** by ordinary construction and induction, ignoring time.

$$\textbf{if } b \textbf{ then } P.\ \textbf{while } b \textbf{ do } P \textbf{ od else } ok \textbf{ fi} \ \Leftarrow\ \textbf{while } b \textbf{ do } P \textbf{ od}$$

$$\forall\sigma, \sigma'\cdot \textbf{ if } b \textbf{ then } P.\, W \textbf{ else } ok \textbf{ fi} \ \Leftarrow\ W \ \Longrightarrow\ \forall\sigma, \sigma'\cdot \textbf{ while } b \textbf{ do } P \textbf{ od} \ \Leftarrow\ W$$

Prove that fixed-point construction and induction

$$\textbf{while } b \textbf{ do } P \textbf{ od} \ =\ \textbf{if } b \textbf{ then } P.\ \textbf{while } b \textbf{ do } P \textbf{ od else } ok \textbf{ fi}$$

$$\forall\sigma, \sigma'\cdot\ W \ =\ \textbf{if } b \textbf{ then } P.\, W \textbf{ else } ok \textbf{ fi} \ \Longrightarrow\ \forall\sigma, \sigma'\cdot \textbf{ while } b \textbf{ do } P \textbf{ od} \ \Leftarrow\ W$$

are theorems.

After trying the question, scroll down to the solution.

§ Start with the ordinary construction axiom, universally quantified

$$\forall\sigma,\sigma'\cdot\ \textbf{if}\ b\ \textbf{then}\ P.\ \ \textbf{while}\ b\ \textbf{do}\ P\ \textbf{od}\ \textbf{else}\ ok\ \textbf{fi}\ \ \Leftarrow\ \ \textbf{while}\ b\ \textbf{do}\ P\ \textbf{od}$$

by monotonicity of sequential composition (stepwise refinement)

$$\Rightarrow\ \forall\sigma,\sigma'\cdot\ (\quad P.\ \ \textbf{if}\ b\ \textbf{then}\ P.\ \ \textbf{while}\ b\ \textbf{do}\ P\ \textbf{od}\ \textbf{else}\ ok\ \textbf{fi}\ \ \Leftarrow\ \ P.\ \ \textbf{while}\ b\ \textbf{do}\ P)$$

by monotonicity of **if** (stepwise refinement)

$$\Rightarrow\ \forall\sigma,\sigma'\cdot\ (\quad \textbf{if}\ b\ \textbf{then}\ P.\ \ \textbf{if}\ b\ \textbf{then}\ P.\ \ \textbf{while}\ b\ \textbf{do}\ P\ \textbf{od}\ \textbf{else}\ ok\ \textbf{fi}\ \textbf{else}\ ok\ \textbf{fi}\ \ \Leftarrow\ \ \textbf{if}\ b\ \textbf{then}\ P.\ \ \textbf{while}\ b\ \textbf{do}\ P\ \textbf{od}\ \textbf{else}\ ok\ \textbf{fi})$$

This is the antecedent of the induction axiom with W replaced by $\textbf{if}\ b\ \textbf{then}\ P.\ \ \textbf{while}\ b\ \textbf{do}\ P\ \textbf{od}\ \textbf{else}\ ok\ \textbf{fi}$, hence we conclude its consequent with the same replacement.

$$\Rightarrow\ \forall\sigma,\sigma'\cdot\ \textbf{while}\ b\ \textbf{do}\ P\ \textbf{od}\ \ \Leftarrow\ \ \textbf{if}\ b\ \textbf{then}\ P.\ \ \textbf{while}\ b\ \textbf{do}\ P\ \textbf{od}\ \textbf{else}\ ok\ \textbf{fi}$$

This is one half of fixed-point construction, and the ordinary construction axiom is the other half. That proves fixed-point construction. As for fixed-point induction, it is immediate from ordinary induction just by strengthening the antecedent. It is interesting to note that ordinary construction and induction can be stated together as one axiom just by strengthening the main connective in induction:

$$\forall\sigma,\sigma'\cdot\ \textbf{if}\ b\ \textbf{then}\ P.\ W\ \textbf{else}\ ok\ \textbf{fi}\ \Leftarrow\ W\ \ =\ \ \forall\sigma,\sigma'\cdot\ \textbf{while}\ b\ \textbf{do}\ P\ \textbf{od}\ \Leftarrow\ W$$