In a graphical program, a pixel might be identified by its Cartesian co-ordinates \(x\) and \(y\), or by its polar co-ordinates \(r\) (radius, or distance from the origin) and \(a\) (angle in radians counter-clockwise from the \(x\) axis). An operation written using one kind of co-ordinates may need to be transformed into the other kind of co-ordinates.

(a) What is the data transformer to transform from Cartesian to polar co-ordinates?

\[ x^2 + y^2 = r^2 \land sin a = y/r \land cos a = x/r \land tan a = y/x \]

This transform has some redundancy; any two of those conjuncts imply the other two. We can already see a constraint on its use: \(r \neq 0 \land x \neq 0\). This constraint has some redundancy: if \(x \neq 0\) then \(r \neq 0\). To transform from Cartesian to polar, this transform is more conveniently written

\[ x = r \times cos a \land y = r \times sin a \]
to use one-point laws to get rid of quantifications \(\forall x, y\) and \(\exists x', y'\).

(b) In Cartesian co-ordinates, one of the operations on a pixel is \(\text{translate}\), which moves a pixel from position \(x\) and \(y\) to position \(x+u\) and \(y+v\).

\[ \text{translate} = x := x+u, y := y+v \]

Use the data transformer from (a) to transform operation \(\text{translate}\) from Cartesian to polar co-ordinates.

\[ \forall x, y \quad x = r \times cos a \land y = r \times sin a \]
\[ \quad \Rightarrow \exists x', y' \quad x' = r' \times cos a' \land y' = r' \times sin a' \land \text{translate} \]
\[ = \forall x, y \quad x = r \times cos a \land y = r \times sin a \]
\[ \quad \Rightarrow \exists x', y' \quad x' = r' \times cos a' \land y' = r' \times sin a' \land (x := x+u, y := y+v) \]
\[ = \forall x, y \quad x = r \times cos a \land y = r \times sin a \]
\[ \quad \Rightarrow \exists x', y' \quad x' = r' \times cos a' \land y' = r' \times sin a' \land x' = x+u \land y' = y+v \]

one-point \(x', y'\)

\[ \forall x, y \quad x = r \times cos a \land y = r \times sin a \]
\[ \quad \Rightarrow r' \times cos a' = x+u \land r' \times sin a' = y+v \]

one-point \(x, y\)

\[ r' \times cos a' = (r \times cos a)+u \land r' \times sin a' = (r \times sin a)+v \]

This is not yet a program. It appears that the way to get a \(\text{translate}\) program in polar co-ordinates is to transform to Cartesian, \(\text{translate}\) in Cartesian, then transform back to polar.

\[ \var x, y: \text{real} \quad x := (r \times cos a)+u, y := (r \times sin a)+v, r := (x^2+y^2)^{1/2}, a := \text{arctan}(y/x) \]

(c) What is the data transformer to transform from polar to Cartesian co-ordinates?

The same transformer from part (a) works, but for this direction, it is more conveniently rewritten, using trigonometric identities, as

\[ r = (x^2+y^2)^{1/2} \land a = \text{arctan}(y/x) \]
to use one-point laws to get rid of quantifications \(\forall r, a\) and \(\exists r', a'\).

(d) In polar co-ordinates, one of the operations on a pixel is \(\text{rotate}\) by \(d\) radians.

\[ \text{rotate} = a := a+d \]

Use the data transformer from (c) to transform operation \(\text{rotate}\) from polar to Cartesian co-ordinates.

\[ \forall r, a \quad r = (x^2+y^2)^{1/2} \land a = \text{arctan}(y/x) \]
\[ \quad \Rightarrow \exists r', a' \quad r' = (x'^2+y'^2)^{1/2} \land a' = \text{arctan}(y'/x') \land \text{rotate} \]
\[ = \forall r, a \quad r = (x^2+y^2)^{1/2} \land a = \text{arctan}(y/x) \]
\[ \quad \Rightarrow \exists r', a' \quad r' = (x'^2+y'^2)^{1/2} \land a' = \text{arctan}(y'/x') \land (a:= a+d) \]
\[ = \forall r, a \quad r = (x^2+y^2)^{1/2} \land a = \text{arctan}(y/x) \]
\[ \quad \Rightarrow \exists r', a' \quad r' = (x'^2+y'^2)^{1/2} \land a' = \text{arctan}(y'/x') \land (r'=r \land a' = a+d) \]

one-point \(r', a'\)
∀r, a·  \[ r = (x^2+y^2)^{1/2} \land a = \arctan (y/x) \]
  \[ (x'^2+y'^2)^{1/2} = r \land \arctan (y'/x') = a+d \]
  \[ (x'^2+y'^2)^{1/2} = (x^2+y^2)^{1/2} \land \arctan (y'/x') = \arctan (y/x) + d \]
  \[ x'^2+y'^2 = x^2+y^2 \land \arctan (y'/x') = \arctan (y/x) + d \]

This is not yet a program. It appears that the way to get a `rotate` program in Cartesian co-ordinates is to transform to polar, `rotate` in polar, then transform back to Cartesian.

\[ \text{var } r, a : \text{real } \]
\[ r := (x^2+y^2)^{1/2} \]
\[ a := \arctan (y/x) + d. \]
\[ x := r \times \cos a. \]
\[ y := r \times \sin a \]