408 Let all variables be integer. Add recursive time. Any way you can, find a fixed-point of

(a) $walk \;=\; \textbf{if } i\ge 0 \textbf{ then } i:= i-2.\; walk.\; i:= i+1.\; walk.\; i:= i+1 \textbf{ else } ok \textbf{ fi}$

(b) $crawl \;=\; \textbf{if } i\ge 0 \textbf{ then } i:= i-1.\; crawl.\; i:= i+2.\; crawl.\; i:= i-1 \textbf{ else } ok \textbf{ fi}$

(c) $run \;=\; \textbf{if } even\ i \textbf{ then } i:= i/2 \textbf{ else } i:= i+1 \textbf{ fi}.\; \textbf{if } i=1 \textbf{ then } ok \textbf{ else } run \textbf{ fi}$

After trying the question, scroll down to the solution.

(a) $walk \;=\; \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-2.\ walk.\ i:= i+1.\ walk.\ i:= i+1\ \mathbf{else}\ ok\ \mathbf{fi}$

§ Putting $t:= t+1$ before just the first call of *walk* is enough, though we could put it before both calls. If the $=$ had been $\Leftarrow$, we could prove $i'=i \;\wedge\; t' \le t+2^i$ as follows. Proof by cases:

$i\ge 0 \;\wedge\; (i:= i-2.\ i'=i \;\wedge\; t' \le t+2^i.\ i:= i+1.\ i'=i \;\wedge\; t' \le t+2^i.\ i:= i+1)$ substitution

$= \quad i\ge 0 \;\wedge\; (i'=i-2 \;\wedge\; t' \le t+2^{i-2}.\ i'=i+1 \;\wedge\; t' \le t+2^{i+1}.\ i:= i+1)$

$= \quad i\ge 0 \;\wedge\; \exists i'', t'', i''', t'''\cdot\ i''=i-2 \wedge t''\le t+2^{i-2} \wedge i'''=i''+1 \wedge t'''\le t''+2^{i''+1} \wedge i'=i'''+1$

$= \quad i\ge 0 \;\wedge\; i'=i \;\wedge\; t' \le t+2^{i-2}+2^{i-1}$

$= \quad i\ge 0 \;\wedge\; i'=i \;\wedge\; t' \le t+(3/4)\times 2^i$

$\Rightarrow \quad i'=i \;\wedge\; t' \le t+2^i$

$i<0 \wedge ok$

$\Rightarrow \quad i'=i \;\wedge\; t' \le t+2^i$

But the question did not say $\Leftarrow$, so I haven't found the fixed-point asked for.

(b) $crawl \;=\; \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ crawl.\ i:= i+2.\ crawl.\ i:= i-1\ \mathbf{else}\ ok\ \mathbf{fi}$

§ Putting $t:= t+1$ before just the first call of *crawl* is enough, though we could put it before both calls. Here are two answers.

$\mathbf{if}\ i\ge 0\ \mathbf{then}\ t'=\infty\ \mathbf{else}\ ok\ \mathbf{fi}$

$\mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= \infty\ \mathbf{else}\ ok\ \mathbf{fi}$

I'll check the second one.

$\mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ t:= t+1.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= \infty\ \mathbf{else}\ ok\ \mathbf{fi}.$
$\quad i:= i+2.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= \infty\ \mathbf{else}\ ok\ \mathbf{fi}.$
$\quad i:= i-1$ distribute $i:= i-1$ back into previous **if**
$\mathbf{else}\ ok\ \mathbf{fi}$

$= \quad \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ t:= t+1.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= \infty\ \mathbf{else}\ ok\ \mathbf{fi}.$
$\quad i:= i+2.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= \infty.\ i:= i-1\ \mathbf{else}\ ok.\ i:= i-1\ \mathbf{fi}$
$\mathbf{else}\ ok\ \mathbf{fi}$ *ok* is identity. Also, expand $i:= i-1$ twice

$= \quad \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ t:= t+1.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= \infty\ \mathbf{else}\ ok\ \mathbf{fi}.$
$\quad i:= i+2.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= \infty.\ i'=i-1 \wedge t'=t\ \mathbf{else}\ i'=i-1 \wedge t'=t\ \mathbf{fi}$
$\mathbf{else}\ ok\ \mathbf{fi}$ substitution law using $t:= \infty$

$= \quad \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ t:= t+1.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= \infty\ \mathbf{else}\ ok\ \mathbf{fi}.$ expand $t:= \infty$ and *ok*
$\quad i:= i+2.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ i'=i-1 \wedge t'=\infty\ \mathbf{else}\ i'=i-1 \wedge t'=t\ \mathbf{fi}$
$\mathbf{else}\ ok\ \mathbf{fi}$

$= \quad \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ t:= t+1.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ i'=i \wedge t'=\infty\ \mathbf{else}\ i'=i \wedge t'=t\ \mathbf{fi}.$ subst law twice
$\quad i:= i+2.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ i'=i-1 \wedge t'=\infty\ \mathbf{else}\ i'=i-1 \wedge t'=t\ \mathbf{fi}$ substitution law
$\mathbf{else}\ ok\ \mathbf{fi}$

$= \quad \mathbf{if}\ i\ge 0\ \mathbf{then}\ \mathbf{if}\ i\ge 1\ \mathbf{then}\ i'=i-1 \wedge t'=\infty\ \mathbf{else}\ i'=i-1 \wedge t'=t+1\ \mathbf{fi}.$ factor (distributive law)
$\quad \mathbf{if}\ i\ge -2\ \mathbf{then}\ i'=i+1 \wedge t'=\infty\ \mathbf{else}\ i'=i+1 \wedge t'=t\ \mathbf{fi}$ factor (distributive law)
$\mathbf{else}\ ok\ \mathbf{fi}$

$= \quad \mathbf{if}\ i\ge 0\ \mathbf{then}\ \mathbf{if}\ i\ge 1\ \mathbf{then}\ t'=\infty\ \mathbf{else}\ t'=t+1\ \mathbf{fi} \wedge i'=i-1.$ definition of .
$\quad \mathbf{if}\ i\ge -2\ \mathbf{then}\ t'=\infty\ \mathbf{else}\ t'=t\ \mathbf{fi} \wedge i'=i+1$
$\mathbf{else}\ ok\ \mathbf{fi}$

$= \quad \mathbf{if}\ i\ge 0\ \mathbf{then}\ \exists i'', t''\cdot\ \mathbf{if}\ i\ge 1\ \mathbf{then}\ t''=\infty\ \mathbf{else}\ t''=t+1\ \mathbf{fi} \wedge i''=i-1$ one-point on i''
$\quad \wedge\ \mathbf{if}\ i''\ge -2\ \mathbf{then}\ t'=\infty\ \mathbf{else}\ t'=t''\ \mathbf{fi} \wedge i'=i''+1$
$\mathbf{else}\ ok\ \mathbf{fi}$

$= \quad \mathbf{if}\ i\ge 0\ \mathbf{then}\ \exists t''\cdot\ \mathbf{if}\ i\ge 1\ \mathbf{then}\ t''=\infty\ \mathbf{else}\ t''=t+1\ \mathbf{fi}$
$\quad \wedge\ \mathbf{if}\ i\ge -1\ \mathbf{then}\ t'=\infty\ \mathbf{else}\ t'=t''\ \mathbf{fi} \wedge i'=i$ context $i\ge 0$
$\mathbf{else}\ ok\ \mathbf{fi}$

$= \quad \mathbf{if}\ i\ge 0\ \mathbf{then}\ \exists t''\cdot\ \mathbf{if}\ i\ge 1\ \mathbf{then}\ t''=\infty\ \mathbf{else}\ t''=t+1\ \mathbf{fi}$
$\quad \wedge\ t'=\infty \wedge i'=i$
$\mathbf{else}\ ok\ \mathbf{fi}$ Now a tricky move. In the inner **then**-part, use context $i\ge 1$.

And in the inner **else**-part, use context $i<1$.

$=$ **if** $i\ge 0$ **then** $\exists t''\cdot$ **if** $i\ge 1$ **then** $t''=$ **if** $i\ge 1$ **then** ∞ **else** $t+1$ **fi**
else $t''=$ **if** $i\ge 1$ **then** ∞ **else** $t+1$ **fi fi**
$\land\ t'=\infty \land i'=i$
else *ok* **fi** — generic case idempotent

$=$ **if** $i\ge 0$ **then** $\exists t''\cdot$ $t''=$ **if** $i\ge 1$ **then** ∞ **else** $t+1$ **fi**
$\land\ t'=\infty \land i'=i$
else *ok* **fi** — one-point for t''

$=$ **if** $i\ge 0$ **then** $t'=\infty \land i'=i$
else *ok* **fi** — assignment

$=$ **if** $i\ge 0$ **then** $t:=\infty$ **else** *ok* **fi**

So it's a fixed-point.

(c) *run* $=$ **if** *even* i **then** $i:=i/2$ **else** $i:=i+1$ **fi**.
if $i=1$ **then** *ok* **else** *run* **fi**

§ Without adding time, $i'=1$ and $i\ge 1\Rightarrow i'=1$ are fixed-points. With time, it's difficult to say a fixed-point since it requires saying the exact execution time. If we had an implication instead of an equation, we could get away with a time bound. Recursive construction just leads to a mess, and isn't helpful. To state the exact execution time, define

$f = \langle i: int\cdot$ **if** $i=2$ **then** 0 **else if** *even* i **then** $1+f(i/2)$ **else** $1+f(i+1)$ **fi fi**$\rangle$

Now we can find the following two fixed-points:

$i'=1\ \land\ t'=t+f\,i$

$(i\ge 1\Rightarrow i'=1)\ \land\ t'=t+f\,i$

although f seems like an unfair trick.