407 Let all variables be integer. Add recursive time. Using recursive construction, find a fixed-point of

(a) *skip* $=$ **if** $i \ge 0$ **then** $i:= i-1$. *skip*. $i:= i+1$ **else** *ok* **fi**

(b) *inc* $=$ *ok* ∨ ($i:= i+1$. *inc*)

(c) *sqr* $=$ **if** $i=0$ **then** *ok* **else** $s:= s + 2 \times i - 1$. $i:= i-1$. *sqr* **fi**

(d) *fac* $=$ **if** $i=0$ **then** $f:= 1$ **else** $i:= i-1$. *fac*. $i:= i+1$. $f:= f \times i$ **fi**

(e) *chs* $=$ **if** $a=b$ **then** $c:= 1$ **else** $a:= a-1$. *chs*. $a:= a+1$. $c:= c \times a/(a-b)$ **fi**

(f) *foo* $=$ **if** $i=0$ **then** $i:= 3$ **else** *foo* **fi**

(g) *bar* $=$ $i:= i-1$. **if** $i=0$ **then** $i:= 3$ **else** *bar*. $i:= 3$ **fi**

After trying the question, scroll down to the solution.

(a) $skip \;=\; \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ \ skip.\ \ i:= i+1\ \mathbf{else}\ ok\ \mathbf{fi}$

§ Adding recursive time,

$skip \;=\; \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ \ t:= t+1.\ \ skip.\ \ i:= i+1\ \mathbf{else}\ ok\ \mathbf{fi}$

$skip_0 \;=\; t' \ge t$

$skip_{n+1} \;=\; \mathbf{if}\ i\ge n\ \mathbf{then}\ t' \ge t+n+1\ \mathbf{else}\ \mathbf{if}\ 0\le i<n\ \mathbf{then}\ t:= t+i+1\ \mathbf{else}\ ok\ \mathbf{fi}\ \mathbf{fi}$

$skip_\infty \;=\; \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= t+i+1\ \mathbf{else}\ ok\ \mathbf{fi}$

To show it's a fixed-point, start with the right side of the definition of $skip$, but substitute $skip_\infty$ in place of $skip$,

$\mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ \ t:= t+1.\ \ \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= t+i+1\ \mathbf{else}\ ok\ \mathbf{fi}.\ \ i:= i+1\ \mathbf{else}\ ok\ \mathbf{fi}$

distribute $i:= i+1$ into preceding **if**

$=\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ \ t:= t+1.\ \ \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= t+i+1.\ i:= i+1\ \mathbf{else}\ ok.\ i:= i+1\ \mathbf{fi}\ \mathbf{else}\ ok\ \mathbf{fi}$

replace first $i:= i+1$ and ok is identity for $.$

$=\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ t:= t+1.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:=t+i+1.\ i'=i+1 \wedge t'=t\ \mathbf{else}\ i:= i+1\ \mathbf{fi}\ \mathbf{else}\ ok\ \mathbf{fi}$

substitution law in second **then**-part

$=\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:= i-1.\ t:= t+1.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ i'=i+1 \wedge t'=t+i+1\ \mathbf{else}\ i:= i+1\ \mathbf{fi}\ \mathbf{else}\ ok\ \mathbf{fi}$

replace $i:= i+1$

$=\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ i:=i-1.\ t:= t+1.\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ i'=i+1 \wedge t'=t+i+1\ \mathbf{else}\ i'=i+1 \wedge t'=t\ \mathbf{fi}\ \mathbf{else}\ ok\ \mathbf{fi}$

substitution law twice more

$=\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ \mathbf{if}\ i-1\ge 0\ \mathbf{then}\ i'=i-1+1 \wedge t'=t+1+i-1+1\ \mathbf{else}\ i'=i-1+1 \wedge t'=t+1\ \mathbf{fi}\ \mathbf{else}\ ok\ \mathbf{fi}$

simplify

$=\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ \mathbf{if}\ i\ge 1\ \mathbf{then}\ i'=i \wedge t'=t+i+1\ \mathbf{else}\ i'=i \wedge t'=t+1\ \mathbf{fi}\ \mathbf{else}\ ok\ \mathbf{fi}$ use $:=$ twice

$=\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ \mathbf{if}\ i\ge 1\ \mathbf{then}\ t:= t+i+1\ \mathbf{else}\ t:= t+1\ \mathbf{fi}\ \mathbf{else}\ ok\ \mathbf{fi}$

In the first **else**-part the context is $i\ge 0 \wedge \neg(i\ge 1)$ which is $i=0$

$=\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ \mathbf{if}\ i\ge 1\ \mathbf{then}\ t:= t+i+1\ \mathbf{else}\ t:= t+i+1\ \mathbf{fi}\ \mathbf{else}\ ok\ \mathbf{fi}$ case idempotent

$=\ \mathbf{if}\ i\ge 0\ \mathbf{then}\ t:= t+i+1\ \mathbf{else}\ ok\ \mathbf{fi}$

and we get $skip_\infty$ again, so it is a fixed-point.

(b) $inc \;=\; ok \vee (i:= i+1.\ \ inc)$

§ Adding recursive time,

$inc \;=\; ok \vee (i:= i+1.\ \ t:= t+1.\ \ inc)$

Now recursive construction. Starting with $\top$,

$inc_0 \;=\; \top$

$inc_1 \;=\; ok \vee (i:= i+1.\ \ t:= t+1.\ \ inc_0)$

$\;=\; ok \vee \top$

$\;=\; \top$

We have converged, and found that $\top$ is a fixed-point. Perhaps we'll get something more interesting if we start with $t'\ge t$.

$inc_0 \;=\; t'\ge t$

$inc_1 \;=\; ok \vee (i:= i+1.\ \ t:= t+1.\ \ inc_0)$

$\;=\; i'=i \wedge t'=t\ \ \vee\ t'\ge t+1$

$inc_2 \;=\; ok \vee (i:= i+1.\ \ t:= t+1.\ \ inc_1)$

$\;=\; i'=i \wedge t'=t\ \ \vee\ i'=i+1 \wedge t'=t+1\ \vee\ t'\ge t+2$

I'm ready to guess

$inc_n \;=\; (\exists m\colon 0,..n\cdot\ i'=i+m \wedge t'=t+m)\ \vee\ t'\ge t+n$

$inc_\infty \;=\; (\exists m\colon nat\cdot\ i'=i+m \wedge t'=t+m)\ \vee\ t'=\infty$

Now I must test inc_∞ to see if it's a fixed-point.

$ok \vee (i:= i+1.\ \ t:= t+1.\ \ inc_\infty)$

$=\ i'=i \wedge t'=t\ \ \vee\ (\exists m\colon nat\cdot\ i'=i+1+m \wedge t'=t+1+m)\ \vee\ t'=\infty$ arithmetic identity

$=\ i'=i+0 \wedge t'=t+0\ \ \vee\ (\exists m\colon nat\cdot\ i'=i+1+m \wedge t'=t+1+m)\ \vee\ t'=\infty$ change of variable

$=\ i'=i+0 \wedge t'=t+0\ \ \vee\ (\exists m\colon nat+1\cdot\ i'=i+m \wedge t'=t+m)\ \vee\ t'=\infty$ basic quantifier law

$=\ (\exists m\colon 0\cdot\ i'=i+m \wedge t'=t+m)\ \ \vee\ (\exists m\colon nat+1\cdot\ i'=i+m \wedge t'=t+m)\ \vee\ t'=\infty$

basic quantifier law

$= \quad (\exists m{:}\ 0, nat{+}1\cdot\ i'{=}i{+}m \wedge t'{=}t{+}m)\ \vee\ t'{=}\infty$ fixed-point nat construction

$= \quad (\exists m{:}\ nat\cdot\ i'{=}i{+}m \wedge t'{=}t{+}m)\ \vee\ t'{=}\infty$

$= \quad inc_\infty$

Starting with $\bot$ we get

$inc_0 \;=\; \bot$

$inc_1 \;=\; ok \vee (i{:=}\ i{+}1.\ t{:=}\ t{+}1.\ inc_0)$

$\quad\;=\; i'{=}i \wedge t'{=}t$

$inc_2 \;=\; ok \vee (i{:=}\ i{+}1.\ t{:=}\ t{+}1.\ inc_1)$

$\quad\;=\; i'{=}i \wedge t'{=}t\ \ \vee\ i'{=}i{+}1 \wedge t'{=}t{+}1$

$inc_n \;=\; (\exists m{:}\ 0,..n\cdot\ i'{=}i{+}m \wedge t'{=}t{+}m)$

$inc_\infty \;=\; (\exists m{:}\ nat\cdot\ i'{=}i{+}m \wedge t'{=}t{+}m)$

This is a fixed-point (not proven here), and it's implementable too (also not proven here)!

(c) $sqr \;=\;$ **if** $i{=}0$ **then** ok **else** $s{:=}\ s + 2{\times}i - 1.\ i{:=}\ i{-}1.\ sqr$ **fi**

§ $sqr_0 \;=\; t'{\ge}t$

$sqr_1 \;=\;$ **if** $i{=}0$ **then** ok **else** $s{:=}\ s + 2{\times}i - 1.\ i{:=}\ i{-}1.\ t{:=}\ t{+}1.\ sqr_0$ **fi**

$\quad=\;$ **if** $i{=}0$ **then** ok **else** $t' \ge t{+}1$

$sqr_2 \;=\;$ **if** $i{=}0$ **then** ok **else** $s{:=}\ s + 2{\times}i - 1.\ i{:=}\ i{-}1.\ t{:=}\ t{+}1.\ sqr_1$ **fi**

$\quad=\;$ **if** $i{=}0$ **then** ok **else** $s{:=}\ s + 2{\times}i - 1.\ i{:=}\ i{-}1.\ t{:=}\ t{+}1.$
  if $i{=}0$ **then** ok **else** $t' \ge t{+}1$ **fi fi**

$\quad=\;$ **if** $i{=}0$ **then** ok
  else if $i{-}1{=}0$ **then** $s{:=}\ s + 2{\times}i - 1.\ i{:=}\ i{-}1.\ t{:=}\ t{+}1$
  else $t' \ge t{+}2$ **fi fi**

$\quad=\;$ **if** $i{=}0$ **then** $s{:=}\ s{+}0.\ i{:=}\ 0.\ t{:=}\ t{+}0$
  else if $i{=}1$ **then** $s{:=}\ s{+}1.\ i{:=}\ 0.\ t{:=}\ t{+}1$
  else $t' \ge t{+}2$ **fi fi**

$sqr_3 \;=\;$ **if** $i{=}0$ **then** ok
  else $s{:=}\ s + 2{\times}i - 1.\ i{:=}\ i{-}1.\ t{:=}\ t{+}1.$
  if $i{=}0$ **then** $s{:=}\ s{+}0.\ i{:=}\ 0.\ t{:=}\ t{+}0$
  else if $i{=}1$ **then** $s{:=}\ s{+}1.\ i{:=}\ 0.\ t{:=}\ t{+}1$
  else $t' \ge t{+}2$ **fi fi fi**

$\quad=\;$ **if** $i{=}0$ **then** ok
  else if $i{=}1$ **then** $s{:=}\ s + 2{\times}i - 1.\ i{:=}\ i{-}1.\ t{:=}\ t{+}1.$
  $s{:=}\ s{+}0.\ i{:=}\ 0.\ t{:=}\ t{+}0$
  else if $i{=}1$ **then** $s{:=}\ s + 2{\times}i - 1.\ i{=}\ i{-}1.\ t{:=}\ t{+}1.$
  $s{:=}\ s{+}1.\ i{:=}\ 0.\ t{:=}\ t{+}1$
  else $s{:=}\ s + 2{\times}i - 1.\ i{:=}\ i{-}1.\ t{:=}\ t{+}1.\ t' \ge t{+}2$ **fi fi fi**

$\quad=\;$ **if** $i{=}0$ **then** $s{:=}\ s{+}0.\ i{:=}\ 0.\ t{:=}\ t{+}0$
  else if $i{=}1$ **then** $s{:=}\ s{+}1.\ i{:=}\ 0.\ t{:=}\ t{+}1$
  else if $i{=}2$ **then** $s{:=}\ s{+}4.\ i{:=}\ 0.\ t{:=}\ t{+}2$
  else $t' \ge t{+}3$ **fi fi fi**

$sqr_n \;=\;$ **if** $0{\le}i{<}n$ **then** $s{:=}\ s{+}i^2.\ t{:=}\ t{+}i.\ i{:=}\ 0$ **else** $t' \ge t{+}n$ **fi**

$sqr_\infty \;=\;$ **if** $0{\le}i$ **then** $s{:=}\ s{+}i^2.\ t{:=}\ t{+}i.\ i{:=}\ 0$ **else** $t'{=}\infty$ **fi**

Now we test to see if sqr_∞ is a fixed-point.

if $i{=}0$ **then** ok **else** $s{:=}\ s + 2{\times}i - 1.\ i{:=}\ i{-}1.\ t{:=}\ t{+}1.$
  if $0{\le}i$ **then** $s{:=}\ s{+}i^2.\ t{:=}\ t{+}i.\ i{:=}\ 0$ **else** $t'{=}\infty$ **fi fi**

$=\;$ **if** $i{=}0$ **then** ok
  else if $0{\le}i{-}1$ **then** $s{:=}\ s + 2{\times}i - 1.\ i{:=}\ i{-}1.\ t{:=}\ t{+}1.$
  $s{:=}\ s{+}i^2.\ t{:=}\ t{+}i.\ i{:=}\ 0$
  else $s{:=}\ s + 2{\times}i - 1.\ i{:=}\ i{-}1.\ t{:=}\ t{+}1.\ t'{=}\infty$ **fi fi**

$=\;$ **if** $i{=}0$ **then** ok
  else if $1{\le}i$ **then** $s{:=}\ s + 2{\times}i - 1 + (i{-}1)^2.\ t{:=}\ t{+}1{+}i{-}1.\ i{:=}\ 0$

$\textbf{else } t'=\infty \textbf{ fi fi}$

$= \textbf{if } i=0 \textbf{ then } s:= s+i^2.\ t:= t+i.\ i:= 0$
$\textbf{else if } 1\le i \textbf{ then } s:= s+i^2.\ t:= t+i.\ i:= 0$
$\textbf{else } t'=\infty \textbf{ fi fi}$

$= sqr_\infty$

(d) $fac = \textbf{if } i=0 \textbf{ then } f:= 1 \textbf{ else } i:= i-1.\ fac.\ i:= i+1.\ f:= f\times i \textbf{ fi}$

§ Adding time,

$fac = \textbf{if } i=0 \textbf{ then } f:= 1 \textbf{ else } i:= i-1.\ t:= t+1.\ fac.\ i:= i+1.\ f:= f\times i \textbf{ fi}$

Recursive construction starting with $t'\ge t$ produces

$fac_n = \textbf{if } 0\le i<n \textbf{ then } f'=i! \land i'=i \land t'=t+i \textbf{ else } t'\ge t+n \textbf{ fi}$

where $i!$ is "i factorial". Replacing n with ∞ produces

$fac_\infty = \textbf{if } 0\le i \textbf{ then } f'=i! \land i'=i \land t'=t+i \textbf{ else } t'=\infty \textbf{ fi}$

Now we see if fac_∞ is a fixed-point. Starting with the right side of the fac equation,

$\textbf{if } i=0 \textbf{ then } f:= 1 \textbf{ else } i:= i-1.\ t:= t+1.\ fac.\ i:= i+1.\ f:= f\times i \textbf{ fi}$ replace fac with fac_∞

$= \textbf{if } i=0 \textbf{ then } f:= 1$ — expand assignment
$\textbf{else } i:= i-1.\ t:= t+1.\ \textbf{if } 0\le i \textbf{ then } f'=i! \land i'=i \land t'=t+i \textbf{ else } t'=\infty \textbf{ fi}.\ i:= i+1.\ f:= f\times i \textbf{ fi}$

combine and expand the final two assignments

$= \textbf{if } i=0 \textbf{ then } f'=1 \land i'=i \land t'=t$ — use **if**-context in **then**-part
$\textbf{else } i:= i-1.\ t:= t+1.\ \textbf{if } 0\le i \textbf{ then } f'=i! \land i'=i \land t'=t+i \textbf{ else } t'=\infty \textbf{ fi}.$
$i' = i+1 \land f' = f\times(i+1) \land t'=t \textbf{ fi}$ — distribute this line into **then** and **else** parts

$= \textbf{if } i=0 \textbf{ then } f'=i! \land i'=i \land t'=t+i$
$\textbf{else } i:= i-1.\ t:= t+1.\ \textbf{if } 0\le i \textbf{ then } f'=i! \land i'=i \land t'=t+i.\ i' = i+1 \land f' = f\times(i+1) \land t'=t$
$\textbf{else } t'=\infty.\ i' = i+1 \land f' = f\times(i+1) \land t'=t \textbf{ fi fi}$ — dep't comp.

$= \textbf{if } i=0 \textbf{ then } f'=i! \land i'=i \land t'=t+i$
$\textbf{else } i:= i-1.\ t:= t+1.\ \textbf{if } 0\le i \textbf{ then } f'=(i+1)! \land i'=i+1 \land t'=t+i \textbf{ else } t'=\infty \textbf{ fi fi}$

substitution law twice

$= \textbf{if } i=0 \textbf{ then } f'=i! \land i'=i \land t'=t+i$
$\textbf{else if } 1\le i \textbf{ then } f'=i! \land i'=i \land t'=t+i \textbf{ else } t'=\infty \textbf{ fi fi}$ — combine $i=0$ and $1\le i$ cases

$= \textbf{if } 0\le i \textbf{ then } f'=i! \land i'=i \land t'=t+i \textbf{ else } t'=\infty \textbf{ fi}$

Therefore fac_∞ is a fixed-point.

(e) $chs = \textbf{if } a=b \textbf{ then } c:= 1 \textbf{ else } a:= a-1.\ chs.\ a:= a+1.\ c:= c\times a/(a-b) \textbf{ fi}$

§ $chs_0 = t'\ge t$

$chs_1 = \textbf{if } a=b \textbf{ then } c:= 1 \textbf{ else } a:= a-1.\ t:= t+1.\ chs_0.\ a:= a+1.\ c:= c\times a/(a-b) \textbf{ fi}$

At this point we need to know that $c\times a/(a-b){:}\ int$ and we don't. But this whole procedure just generates a candidate that needs to be tested. So we carry on as if $c\times a/(a-b){:}\ int$

$= \textbf{if } a=b \textbf{ then } c:= 1 \textbf{ else } t' \ge t+1 \textbf{ fi}$

$chs_2 = \textbf{if } a=b \textbf{ then } c:= 1 \textbf{ else } a:= a-1.\ t:= t+1.\ chs_1.\ a:= a+1.\ c:= c\times a/(a-b) \textbf{ fi}$

$= \textbf{if } a=b \textbf{ then } c:= 1$
$\textbf{else } a:= a-1.\ t:= t+1.\ \textbf{if } a=b \textbf{ then } c:= 1 \textbf{ else } t' \ge t+1 \textbf{ fi}.$
$a:= a+1.\ c:= c\times a/(a-b) \textbf{ fi}$

$= \textbf{if } a=b \textbf{ then } c:= 1$
$\textbf{else if } a-1=b \textbf{ then } a:= a-1.\ t:= t+1.\ c:= 1.\ a:= a+1.\ c:= c\times a/(a-b)$
$\textbf{else } a:= a-1.\ t:= t+1.\ t' \ge t+1.\ a:= a+1.\ c:= c\times a/(a-b) \textbf{ fi fi}$

$= \textbf{if } a=b \textbf{ then } c:= 1$
$\textbf{else if } a-1=b \textbf{ then } t:= t+1.\ c:= a$
$\textbf{else } t' \ge t+2 \textbf{ fi fi}$

$chs_3 = \textbf{if } a=b \textbf{ then } c:= 1$
$\textbf{else } a:= a-1.\ t:= t+1.$
$\textbf{if } a=b \textbf{ then } c:= 1$

$\mathbf{else\ if}\ a{-}1{=}b\ \mathbf{then}\ t{:=}\ t{+}1.\ \ c{:=}\ a$

$\mathbf{else}\ t' \ge t{+}2\ \mathbf{fi\ fi}.$

$a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)\ \mathbf{fi}$

$=\ \ \mathbf{if}\ a{=}b\ \mathbf{then}\ c{:=}\ 1$

$\mathbf{else\ if}\ a{-}1{=}b\ \mathbf{then}\ a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.\ \ c{:=}\ 1.\ \ a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)$

$\mathbf{else\ if}\ a{-}2{=}b\ \mathbf{then}\ a{:=}\ a{-}1.\ t{:=}\ t{+}1.\ t{:=}\ t{+}1.\ c{:=}\ a.\ a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)$

$\mathbf{else}\ a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.\ \ t' \ge t{+}2.\ \ a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)\ \mathbf{fi\ fi\ fi}$

$=\ \ \mathbf{if}\ a{=}b\ \mathbf{then}\ c{:=}\ 1$

$\mathbf{else\ if}\ a{-}1{=}b\ \mathbf{then}\ t{:=}\ t{+}1.\ \ c{:=}\ a$

$\mathbf{else\ if}\ a{-}2{=}b\ \mathbf{then}\ t{:=}\ t{+}2.\ \ c{:=}\ a{\times}(a{-}1)/2$

$\mathbf{else}\ t' \ge t{+}3\ \mathbf{fi\ fi\ fi}$

$chs_4\ =\ \ \mathbf{if}\ a{=}b\ \mathbf{then}\ c{:=}\ 1$

$\mathbf{else}\ \ a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.$

$\mathbf{if}\ a{=}b\ \mathbf{then}\ c{:=}\ 1$

$\mathbf{else\ if}\ a{-}1{=}b\ \mathbf{then}\ t{:=}\ t{+}1.\ \ c{:=}\ a$

$\mathbf{else\ if}\ a{-}2{=}b\ \mathbf{then}\ t{:=}\ t{+}2.\ \ c{:=}\ a{\times}(a{-}1)/2$

$\mathbf{else}\ t' \ge t{+}3\ \mathbf{fi\ fi\ fi}.$

$a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)\ \mathbf{fi}$

$=\ \ \mathbf{if}\ a{=}b\ \mathbf{then}\ c{:=}\ 1$

$\mathbf{else\ if}\ a{-}1{=}b\ \mathbf{then}\ t{:=}\ t{+}1.\ \ c{:=}\ a$

$\mathbf{else\ if}\ a{-}2{=}b\ \mathbf{then}\ t{:=}\ t{+}2.\ \ c{:=}\ a{\times}(a{-}1)/2$

$\mathbf{else\ if}\ a{-}3{=}b\ \mathbf{then}\ t{:=}\ t{+}3.\ \ c{:=}\ a{\times}(a{-}1){\times}(a{-}2)/(2{\times}3)$

$\mathbf{else}\ t' \ge t{+}4\ \mathbf{fi\ fi\ fi\ fi}$

$chs_n\ =\ \ \mathbf{if}\ b \le a < b{+}n\ \mathbf{then}\ t{:=}\ t{+}a{-}b.\ c{:=}\ \Pi[b{+}1;..a{+}1]/\Pi[1;..a{-}b{+}1])\ \mathbf{else}\ t' \ge t{+}n\ \mathbf{fi}$

$chs_\infty\ =\ \ \mathbf{if}\ a{\ge}b\ \mathbf{then}\ t{:=}\ t{+}a{-}b.\ \ c{:=}\ \Pi[b{+}1;..a{+}1]/\Pi[1;..a{-}b{+}1]\ \mathbf{else}\ t'{=}\infty\ \mathbf{fi}$

Now I test to see if chs_∞ is a fixed-point.

$\mathbf{if}\ a{=}b\ \mathbf{then}\ c{:=}\ 1\ \mathbf{else}\ a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.\ \ chs_\infty.\ \ a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)\ \mathbf{fi}$

$=\ \ \mathbf{if}\ a{=}b\ \mathbf{then}\ c{:=}\ 1$

$\mathbf{else}\ \ a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.$

$\mathbf{if}\ a{\ge}b\ \mathbf{then}\ t{:=}\ t{+}a{-}b.\ c{:=}\ \Pi[b{+}1;..a{+}1]/\Pi[1;..a{-}b{+}1]\ \mathbf{else}\ t'{=}\infty\ \mathbf{fi}.$

$a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)\ \mathbf{fi}$

$=\ \ \mathbf{if}\ a{=}b\ \mathbf{then}\ c{:=}\ 1$

$\mathbf{else\ if}\ a{-}1{\ge}b\ \mathbf{then}\ \ a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.$

$t{:=}\ t{+}a{-}b.\ \ c{:=}\ \Pi[b{+}1;..a{+}1]/\Pi[1;..a{-}b{+}1].$

$a{:=}\ a{+}1.\ c{:=}\ c{\times}a/(a{-}b)$

$\mathbf{else}\ a{:=}\ a{-}1.\ \ t{:=}\ t{+}1.\ \ t'{=}\infty.\ \ a{:=}\ a{+}1.\ \ c{:=}\ c{\times}a/(a{-}b)\ \mathbf{fi\ fi}$

$=\ \ \mathbf{if}\ a{=}b\ \mathbf{then}\ c{:=}\ 1$

$\mathbf{else\ if}\ a{>}b\ \mathbf{then}\ t{:=}\ t{+}a{-}b.\ \ c{:=}\ \Pi[b{+}1;..a{+}1]/\Pi[1;..a{-}b{+}1]$

$\mathbf{else}\ t'{=}\infty\ \mathbf{fi\ fi}$

$=\ \ \mathbf{if}\ a{\ge}b\ \mathbf{then}\ t{:=}\ t{+}a{-}b.\ \ c{:=}\ \Pi[b{+}1;..a{+}1]/\Pi[1;..a{-}b{+}1]\ \mathbf{else}\ t'{=}\infty\ \mathbf{fi}$

$=\ \ chs_\infty$

So chs_∞ is a fixed-point. Note that for $1{\le}b{\le}a$, c' is the number of ways of choosing b things from a things.

(f) $\quad foo\ =\ \mathbf{if}\ i{=}0\ \mathbf{then}\ i{:=}\ 3\ \mathbf{else}\ foo\ \mathbf{fi}$

§ $\quad foo_0\ =\ \top$

$foo_1\ =\ \mathbf{if}\ i{=}0\ \mathbf{then}\ i{:=}\ 3\ \mathbf{else}\ t{:=}\ t{+}1.\ \ \top\ \mathbf{fi}$

$=\ \mathbf{if}\ i{=}0\ \mathbf{then}\ i'{=}3 \land t'{=}t\ \mathbf{else}\ \top\ \mathbf{fi}$

$=\ i{=}0 \Rightarrow i'{=}3 \land t'{=}t$

$foo_2\ =\ \mathbf{if}\ i{=}0\ \mathbf{then}\ i{:=}\ 3\ \mathbf{else}\ t{:=}\ t{+}1.\ \ i{=}0 \Rightarrow i'{=}3 \land t'{=}t\ \mathbf{fi}$

$=\ \mathbf{if}\ i{=}0\ \mathbf{then}\ i'{=}3 \land t'{=}t\ \mathbf{else}\ i{=}0 \Rightarrow i'{=}3 \land t'{=}t{+}1\ \mathbf{fi}$ context

$=\ \mathbf{if}\ i{=}0\ \mathbf{then}\ i'{=}3 \land t'{=}t\ \mathbf{else}\ \top\ \mathbf{fi}$

$= \quad i{=}0 \Rightarrow i'{=}3 \wedge t'{=}t$

$= \quad foo_1$

The weakest fixed-point (solution) $i{=}0 \Rightarrow i'{=}3 \wedge t'{=}t$ has been found.

(g) $bar \quad = \quad i{:=}\ i{-}1.\ \ \mathbf{if}\ i{=}0\ \mathbf{then}\ i{:=}\ 3\ \mathbf{else}\ bar.\ \ i{:=}\ 3\ \mathbf{fi}$

§ $bar_0 \quad = \quad \top$

$bar_1 \quad = \quad i{:=}\ i{-}1.\ \ \mathbf{if}\ i{=}0\ \mathbf{then}\ i{:=}\ 3\ \mathbf{else}\ t{:=}\ t{+}1.\ \ \top.\ \ i{:=}\ 3\ \mathbf{fi}$

$= \quad i{:=}\ i{-}1.\ \ \mathbf{if}\ i{=}0\ \mathbf{then}\ i'{=}3 \wedge t'{=}t\ \mathbf{else}\ t{:=}\ t{+}1.\ \ \top.\ \ i'{=}3 \wedge t'{=}t\ \mathbf{fi}$

$= \quad i{:=}\ i{-}1.\ \ \mathbf{if}\ i{=}0\ \mathbf{then}\ i'{=}3 \wedge t'{=}t\ \mathbf{else}\ t{:=}\ t{+}1.\ \ (\exists i'', t''\cdot\ \top \wedge i'{=}3 \wedge t'{=}t'')\ \mathbf{fi}$

$= \quad i{:=}\ i{-}1.\ \ \mathbf{if}\ i{=}0\ \mathbf{then}\ i'{=}3 \wedge t'{=}t\ \mathbf{else}\ t{:=}\ t{+}1.\ \ i'{=}3\ \mathbf{fi}$

$= \quad i{:=}\ i{-}1.\ \ \mathbf{if}\ i{=}0\ \mathbf{then}\ i'{=}3 \wedge t'{=}t\ \mathbf{else}\ i'{=}3\ \mathbf{fi}$

$= \quad \mathbf{if}\ i{=}1\ \mathbf{then}\ i'{=}3 \wedge t'{=}t\ \mathbf{else}\ i'{=}3\ \mathbf{fi}$

$= \quad i'{=}3 \wedge \mathbf{if}\ i{=}1\ \mathbf{then}\ t'{=}t\ \mathbf{else}\ \top\ \mathbf{fi}$

$= \quad i'{=}3 \wedge (i{=}1 \Rightarrow t'{=}t)$

$bar_2 \quad = \quad i{:=}\ i{-}1.\ \ \mathbf{if}\ i{=}0\ \mathbf{then}\ i{:=}\ 3\ \mathbf{else}\ t{:=}\ t{+}1.\ \ i'{=}3 \wedge (i{=}1 \Rightarrow t'{=}t).\ \ i{:=}\ 3\ \mathbf{fi}$

$= \quad i{:=}\ i{-}1.\ \ \mathbf{if}\ i{=}0\ \mathbf{then}\ i'{=}3 \wedge t'{=}t\ \mathbf{else}\ i'{=}3 \wedge (i{=}1 \Rightarrow t'{=}t{+}1)\ \mathbf{fi}$

$= \quad \mathbf{if}\ i{=}1\ \mathbf{then}\ i'{=}3 \wedge t'{=}t\ \mathbf{else}\ i'{=}3 \wedge (i{=}2 \Rightarrow t'{=}t{+}1)\ \mathbf{fi}$

$= \quad i'{=}3 \wedge \mathbf{if}\ i{=}1\ \mathbf{then}\ t'{=}t\ \mathbf{else}\ i{=}2 \Rightarrow t'{=}t{+}1\ \mathbf{fi}$

$= \quad i'{=}3 \wedge (0{<}i{\le}2 \Rightarrow t'{=}t{+}i{-}1)$

Now I guess

$bar_n \quad = \quad i'{=}3 \wedge (0{<}i{\le}n \Rightarrow t'{=}t{+}i{-}1)$

Replacing n with ∞ produces

$bar_\infty \quad = \quad i'{=}3 \wedge (0{<}i \Rightarrow t'{=}t{+}i{-}1)$

For proof that this is a solution, see Exercise 406(b).