Let all variables be integer. Add recursive time. Using recursive construction, find a fixed-point of

\( \text{skip} \)  
\[ \text{if } i \geq 0 \text{ then } i := i-1. \ \text{skip. } i := i+1 \text{ else } \text{ok fi} \]

Adding recursive time,

\( \text{skip} \)  
\[ \text{if } i \geq 0 \text{ then } i := i-1. \ \text{t := t+1. } \text{skip. } i := i+1 \text{ else } \text{ok fi} \]
\( \text{skip}_0 \)  
\[ t \geq t \]
\( \text{skip}_{n+1} \)  
\[ \text{if } i \geq n \text{ then } i' \geq t+n+1 \text{ else if } 0 \leq i < n \text{ then } t := t+i+1 \text{ else } \text{ok fi fi} \]
\( \text{skip}_\infty \)  
\[ \text{if } i \geq 0 \text{ then } t := t+i+1 \text{ else } \text{ok fi} \]

To show it’s a fixed-point, start with the right side of the definition of \( \text{skip} \), but substitute \( \text{skip}_\infty \) in place of \( \text{skip} \),

\[ \text{if } i \geq 0 \text{ then } i := i-1. \ t := t+1. \ \text{if } i \geq 0 \text{ then } t := t+i+1 \text{ else } \text{ok fi. } i := i+1 \text{ else } \text{ok fi} \]

\[ \text{distribute } i := i+1 \text{ into preceding if} \]
\[ \text{if } i \geq 0 \text{ then } i := i-1. \ t := t+1. \ \text{if } i \geq 0 \text{ then } t := t+i+1. \ i := i+1 \text{ else } \text{ok. } i := i+1 \text{ else } \text{ok fi fi} \]

\[ \text{replace first } i := i+1 \text{ and } \text{ok is identity for} \]
\[ \text{if } i \geq 0 \text{ then } i := i-1. \ t := t+1. \ \text{if } i \geq 0 \text{ then } t := t+i+1. \ i' := i+1 \land t' := t \text{ else } \text{i := i+1 \text{ else } \text{ok fi fi} substitution law in second then-part} \]
\[ \text{if } i \geq 0 \text{ then } i := i-1. \ t := t+1. \ \text{if } i \geq 0 \text{ then } t := t+i+1 \text{ else } i := i+1 \text{ else } \text{ok fi fi} \]

\[ \text{substitution law twice more} \]
\[ \text{if } i \geq 0 \text{ then if } i-1 \geq 0 \text{ then } i' := i-1+1 \land t' := t+i+1+i-1+1 \text{ else } i' := i-1+1 \land t' := t+i+1 \text{ else } \text{ok fi fi} \]

\[ \text{simplify} \]
\[ \text{if } i \geq 0 \text{ then if } i \geq 1 \text{ then } i' := i \land t' := t+i+1 \text{ else } i' := i \land t' := t+i+1 \text{ else } \text{ok fi fi} \]

\[ \text{simplify} \]
\[ \text{use } := \text{ twice} \]
\[ \text{case idempotent} \]
\[ \text{if } i \geq 0 \text{ then if } i \geq 1 \text{ then } t := t+i+1 \text{ else } t := t+i+1 \text{ else } \text{ok fi} \]

and we get \( \text{skip}_\infty \) again, so it is a fixed-point.

(b)  
\( \text{inc} \)  
\[ \text{ok v (i := i+1. inc)} \]

Adding recursive time,

\( \text{inc} \)  
\[ \text{ok v (i := i+1. t := t+1. inc)} \]

Now recursive construction. Starting with \( \top \),

\( \text{inc}_0 \)  
\[ \top \]
\( \text{inc}_1 \)  
\[ \text{ok v (i := i+1. t := t+1. inc}_0) \]
\[ = \text{ ok v } \top \]
\[ = \top \]

We have converged, and found that \( \top \) is a fixed-point. Perhaps we’ll get something more interesting if we start with \( t \geq t \).

\( \text{inc}_0 \)  
\[ t' \geq t \]
\( \text{inc}_1 \)  
\[ \text{ok v (i := i+1. t := t+1. inc}_0) \]
\[ = i' = i \land t' = t \lor t' \geq t+1 \]
\( \text{inc}_2 \)  
\[ \text{ok v (i := i+1. t := t+1. inc}_1) \]
\[ = i' = i \land t' = t \lor i' = i+1 \land t' = t+1 \lor t' \geq t+2 \]

I’m ready to guess

\( \text{inc}_n \)  
\[ (\exists m: 0 \ldots n \bullet i' = i+m \land t' = t+m) \lor t' \geq t+n \]
\( \text{inc}_\infty \)  
\[ (\exists m: \text{nat} \bullet i' = i+m \land t' = t+m) \lor t' = \infty \]

Now I must test \( \text{inc}_\infty \) to see if it’s a fixed-point.

\( \text{ok v (i := i+1. t := t+1. inc}_\infty) \]
\[ = i' = i \land t' = t \lor (\exists m: \text{nat} \bullet i' = i+1+m \land t' = t+1+m) \lor t' = \infty \]

arithmetic identity
Now we test to see if \( \text{sqr} \) and it is a fixed-point, and it's implementable too!

\[
\text{sqr} = \begin{cases} 
\text{if } i=0 \text{ then } \text{ok} & \text{else } s := s + 2 \times i - 1. \ i := i-1. \ \text{sqr} \ \text{fi} \\
\text{else if } i=0 \text{ then } \text{ok} & \text{else } s := s + 2 \times i - 1. \ i := i-1. \ t := t+1 \\
\text{else } t' \geq t+2 & \text{fi fi} \\
\end{cases}
\]

\[
\text{sqr}_0 = t' \geq t \\
\text{sqr}_1 = \begin{cases} 
\text{if } i=0 \text{ then } \text{ok} & \text{else } s := s + 2 \times i - 1. \ i := i-1. \ t := t+1 \\
\text{else } t' \geq t+2 & \text{fi fi} \\
\end{cases}
\]

\[
\text{sqr}_2 = \begin{cases} 
\text{if } i=0 \text{ then } \text{ok} & \text{else } s := s + 2 \times i - 1. \ i := i-1. \ t := t+1 \\
\text{else } t' \geq t+2 & \text{fi fi} \\
\end{cases}
\]

\[
\text{sqr}_3 = \begin{cases} 
\text{if } i=0 \text{ then } \text{ok} & \text{else } s := s + 2 \times i - 1. \ i := i-1. \ t := t+1 \\
\text{else if } i=0 \text{ then } \text{ok} & \text{else } s := s+0. \ i := 0. \ t := t+0 \\
\text{else if } i=1 \text{ then } s := s+1. \ i := 0. \ t := t+1 \\
\text{else } t' \geq t+2 & \text{fi fi fi fi} \\
\end{cases}
\]

\[
\text{sqr}_n = \begin{cases} 
\text{if } 0 \leq i < n \text{ then } s := s+i^2. \ t := t+i. \ i := 0 \text{ else } t' \geq t+n & \text{fi} \\
\text{else if } i=0 \text{ then } s := s+0. \ i := 0. \ t := t+0 \\
\text{else if } i=1 \text{ then } s := s+1. \ i := 0. \ t := t+1 \\
\text{else if } i=2 \text{ then } s := s+4. \ i := 0. \ t := t+2 \\
\text{else } t' \geq t+3 \text{ fi fi fi fi} \\
\end{cases}
\]

\[
\text{sqr}_\infty = \begin{cases} 
\text{if } 0 \leq i < n \text{ then } s := s+i^2. \ t := t+i. \ i := 0 \text{ else } t' \geq t+n & \text{fi} \\
\text{else if } i=0 \text{ then } s := s+0. \ i := 0. \ t := t+0 \\
\text{else if } i=1 \text{ then } s := s+1. \ i := 0. \ t := t+1 \\
\text{else if } i=2 \text{ then } s := s+4. \ i := 0. \ t := t+2 \\
\text{else } t' \geq t+3 \text{ fi fi fi fi} \\
\end{cases}
\]

Now we test to see if \( \text{sqr}_\infty \) is a fixed-point.

\[
\text{if } i=0 \text{ then } \text{ok} \text{ else } s := s + 2 \times i - 1. \ i := i-1. \ t := t+1. \\
\text{else if } 0 \leq i-1 \text{ then } s := s + 2 \times i - 1. \ i := i-1. \ t := t+1. \\
\]

\[
\text{if } i=0 \text{ then } \text{ok} \text{ else } s := s + 2 \times i - 1. \ i := i-1. \ t := t+1. \\
\text{else if } 0 \leq i \text{ then } s := s+i^2. \ t := t+i. \ i := 0 \text{ else } t' \geq t+\infty & \text{fi fi fi fi} \\
\text{else if } i=0 \text{ then } \text{ok} \\
\text{else if } 0 \leq i-1 \text{ then } s := s + 2 \times i - 1. \ i := i-1. \ t := t+1. \\
\]

\[
\text{else if } i=0 \text{ then } \text{ok} \text{ else } s := s + 2 \times i - 1. \ i := i-1. \ t := t+1. \\
\text{else if } i=0 \text{ then } \text{ok} \\
\text{else if } 0 \leq i \text{ then } s := s+i^2. \ t := t+i. \ i := 0 \text{ else } t' \geq t+\infty & \text{fi fi fi fi} \\
\text{else if } i=0 \text{ then } \text{ok} \\
\text{else if } 0 \leq i-1 \text{ then } s := s + 2 \times i - 1. \ i := i-1. \ t := t+1. \\
\]
\[ s := s + i^2. \quad t := t + i. \quad i := 0 \]
\[ \text{else } s := s + 2xi - 1. \quad i := i - 1. \quad t := t + 1. \quad t' = \infty \]

\[ \text{if } i = 0 \text{ then } ok \]
\[ \text{else if } 1 \leq i \text{ then } s := s + 2xi - 1 + (i - 1)^2. \quad t := t + 1 + i. \quad i := 0 \]
\[ \text{else } t' = \infty \]

\[ \text{if } i = 0 \text{ then } s := s + i^2. \quad t := t + i. \quad i := 0 \]
\[ \text{else if } 1 \leq i \text{ then } s := s + i^2. \quad t := t + i. \quad i := 0 \]
\[ \text{else } t' = \infty \]
\[ = \quad \text{sqr}_{\infty} \]

(d) \[ \text{fac } \begin{array}{c} = \text{if } i = 0 \text{ then } f := 1 \text{ else } i := i - 1. \quad \text{fac. } \quad i := i + 1. \quad f := f \times i \text{ if } \text{replace fac with } \text{fac}_{\infty} \\
\text{else if } 1 \leq i \text{ then } t := t + 1. \quad \text{fac. } \quad i := i + 1. \quad f := f \times i \text{ if } \text{combine and expand the final two assignments} \\
\text{else } i := i - 1. \quad t := t + 1. \quad \text{if } 0 \leq i \text{ then } f := i! \land i' = i \land t' = t + i \text{ else } t' = \infty \text{ if } \text{use if-context in then-part} \\
\text{else } i := i - 1. \quad t := t + 1. \quad \text{if } 0 \leq i \text{ then } f := i! \land i' = i \land t' = t + i \text{ else } t' = \infty \text{ if } \text{combine } i = 0 \text{ and } 1 \leq i \text{ cases} \\
\text{else } i := i - 1. \quad t := t + 1. \quad \text{if } 0 \leq i \text{ then } f := (i+1)! \land i' = i + 1 \land t' = t + i \text{ else } t' = \infty \text{ if } \text{substitution law twice} \\
\text{else if } 1 \leq i \text{ then } f := i! \land i' = i \land t' = t + i \text{ else } t' = \infty \text{ fi fi} \\
\text{else } 0 \leq i \text{ then } f := i! \land i' = i \land t' = t + i \text{ else } t' = \infty \text{ fi fi} \end{array} \]

Therefore \text{fac}_{\infty} is a fixed-point.

(e) \[ \text{chs } \begin{array}{c} = \text{if } a = b \text{ then } c := 1 \text{ else } a := a - 1. \quad \text{chs. } \quad a := a + 1. \quad c := c \times a / (a - b) \text{ fi} \\
\text{else if } 1 \leq i \text{ then } t := t + 1. \quad \text{chs}_{0}. \quad a := a + 1. \quad c := c \times a / (a - b) \text{ fi} \\
\text{else if } 1 \leq i \text{ then } c := 1 \text{ else } a := a - 1. \quad t := t + 1. \quad \text{chs}_{1}. \quad a := a + 1. \quad c := c \times a / (a - b) \text{ fi} \\
\text{else if } a - 1 = b \text{ then } a := a - 1. \quad t := t + 1. \quad c := 1. \quad a := a + 1. \quad c := c \times a / (a - b) \text{ fi} \\
\text{else if } a - 1 = b \text{ then } c := 1 \text{ else } a := a - 1. \quad t := t + 1. \quad \text{t' \geq t + 1. } \quad a := a + 1. \quad c := c \times a / (a - b) \text{ fi fi} \\
\text{else if } a - 1 = b \text{ then } t := t + 1. \quad c := a \end{array} \]

At this point we need to know that \text{cxl(a-b)}: \text{int} and we don't.

But this whole procedure just generates a candidate that needs to be tested.

So we carry on as if \text{cxl(a-b)}: \text{int}
Now I test to see if $chs_3$ is a fixed-point.

```plaintext
chs_3 =
  if \(a=b\) then \(c:=1\)
  else \(a:=a-1. \ t:=t+1.\)
    if \(a=b\) then \(c:=1\)
    else if \(a-1=b\) then \(t:=t+1. \ c:=a\)
      else \(t' \geq t+2 \text{ fi fi}\).
    \(a:=a+1. \ c:=c\times a/(a-b) \text{ fi fi}\)
  if \(a=b\) then \(c:=1\)
  else if \(a-1=b\) then \(a:=a-1. \ t:=t+1. \ c:=a+1. \ c:=c\times a/(a-b)\)
    else if \(a-2=b\) then \(a:=a-1. \ t:=t+1. \ c:=a. \ a:=a+1. \ c:=c\times a/(a-b)\)
      else \(a-1. \ t:=t+1. \ t' \geq t+2. \ a:=a+1. \ c:=c\times a/(a-b) \text{ fi fi fi}\)
  if \(a=b\) then \(c:=1\)
  else if \(a-1=b\) then \(t:=t+1. \ c:=a\)
    else if \(a-2=b\) then \(t:=t+2. \ c:=a\times (a-1)/2\)
      else \(t' \geq t+3 \text{ fi fi fi}\).
  \(a:=a+1. \ c:=c\times a/(a-b) \text{ fi fi}\)
=  if \(a=b\) then \(c:=1\)
  else if \(a-1=b\) then \(t:=t+1. \ c:=a\)
    else if \(a-2=b\) then \(t:=t+2. \ c:=a\times (a-1)/2\)
      else if \(a-3=b\) then \(t:=t+3. \ c:=a\times a/(a-2)/(2\times 3)\)
        else \(t' \geq t+4 \text{ fi fi fi fi}\)
  \(a:=a+1. \ c:=c\times a/(a-b) \text{ fi fi}\)
=  \(a \leq a < b+n \text{ then t:=t+a-b. c:=}\Pi[b+1;..a+1]/\Pi[1;..a-b+1]\) else \(t' \geq t+n \text{ fi fi}\)
=  \(a \geq b \text{ then t:=t+a-b. c:=}\Pi[b+1;..a+1]/\Pi[1;..a-b+1]\) else \(t'=\infty \text{ fi fi}\)
Now I test to see if $chs_n$ is a fixed-point.

```plaintext
\(a=b \text{ then c:=1 else a:=a-1. t:=t+1. chs_n. a:=a+1. c:=c\times a/(a-b) \text{ fi fi}\)
=  \(a=b \text{ then c:=1 else a-1 \geq b \text{ then a:=a-1. t:=t+1. t:=t+a-b. c:=}\Pi[b+1;..a+1]/\Pi[1;..a-b+1].
\ a:=a+1. c:=c\times a/(a-b) \text{ fi fi}\)
\(a=b \text{ then c:=1 else a\geq b \text{ then a:=a-1. t:=t+1. t'=}\infty. a:=a+1. c:=c\times a/(a-b) \text{ fi fi}\)
=  \(a=b \text{ then c:=1 else a\geq b \text{ then t:=t+a-b. c:=}\Pi[b+1;..a+1]/\Pi[1;..a-b+1]\) else \(t'=\infty \text{ fi fi}\)
=  \(chs_n\)
Now I test to see if $chs_n$ is a fixed-point. Note that for $1 \leq b \leq a$, $c'$ is the number of ways of choosing $b$ things from $a$ things.

\(\text{foo} = \text{if } i=0 \text{ then } i:=3 \text{ else foo fi}\)

\(\text{foo}_0 = T\)
\(\text{foo}_1 = \text{if } i=0 \text{ then } i:=3 \text{ else } t:=t+1. \ T \text{ fi}\)
\(= \text{if } i=0 \text{ then } i'=3 \wedge t'=t \text{ else } T \text{ fi}\)
\[ i = 0 \Rightarrow i' = 3 \land t' = t \]

\[
\text{foo}_2 = \begin{cases} 
  i := 3 & \text{if } i = 0 \\
  i := i' & \text{else} 
\end{cases} \land t := t+1.
\]

\[
i = 0 \Rightarrow i' = 3 \land t' = t \land \text{context}
\]

The weakest fixed-point (solution) \( i = 0 \Rightarrow i' = 3 \land t' = t \) has been found.

\[
(g) \quad \text{bar} = \begin{cases} 
  i := i-1 & \text{if } i = 0 \\
  i := 3 & \text{else} 
\end{cases} \land \text{bar}.
\]

\[
\text{§} \quad \text{bar}_0 = \top
\]

\[
\begin{aligned}
\text{bar}_1 &= \begin{cases} 
  i := i-1 & \text{if } i = 0 \\
  i := 3 & \text{else}
\end{cases} \land \text{bar}.
\end{aligned}
\]

\[
\begin{aligned}
\text{bar}_2 &= \begin{cases} 
  i := i-1 & \text{if } i = 0 \\
  i := 3 & \text{else}
\end{cases} \land (i = 1 \Rightarrow t' = t).
\end{aligned}
\]

Now I guess

\[
\begin{aligned}
\text{bar}_n &= i' = 3 \land (0 < i \leq n \Rightarrow t' = t + i - 1) \\
\text{bar}_\infty &= i' = 3 \land (0 < i \Rightarrow t' = t + i - 1)
\end{aligned}
\]