(bit strings) Let \( a, b, c: *(0, 1) \). Define operator \( \oplus \) (precedence 4) as follows.

\[
\begin{align*}
\text{nil} \oplus \text{nil} &= 0 \\
\text{a} \oplus 0 &= \text{a} \\
(a; 0) \oplus 1 &= a; 1 \\
(a; 1) \oplus 1 &= a \oplus 1; 0 \\
a \oplus b &= b \oplus a \\
(a \oplus b) \oplus c &= a \oplus (b \oplus c)
\end{align*}
\]

Prove

(a) \( a \oplus a = a; 0 \)
(b) \( a; 0 = a \oplus a \oplus 0 \)
(c) \( a; 1 = a \oplus a \oplus 1 \)

After trying the question, scroll down to the start of a solution.
§
(a) \[ a \oplus a = a; 0 \]
Since \( a : *(0, 1) \) then \( a \) is one of \( \text{nil} \) or \( b;0 \) or \( b;1 \) for some \( b : *(0, 1) \).
Case \( a = \text{nil} \)
\[ a \oplus a \]
\[ = \text{nil} \oplus \text{nil} \]
\[ = 0 \]
\[ = \text{nil}; 0 \]
\[ = a; 0 \]
Case \( a = b; 0 \)
\[ a \oplus a \]
\[ = \text{UNFINISHED} \]
\[ = a; 0 \]
Case \( a = b; 1 \)
\[ a \oplus a \]
\[ = \text{UNFINISHED} \]
\[ = a; 0 \]

That was an induction, but it was not fully formal because we don't have the induction axiom for \( * \).

(b) \[ a; 0 = a \oplus a \oplus 0 \]
\[ a \oplus a \oplus 0 \]
\[ = a \oplus a \]
\[ = a; 0 \]

(c) \[ a; 1 = a \oplus a \oplus 1 \]
\[ a \oplus a \oplus 1 \]
\[ = \text{UNFINISHED} \]
\[ = a; 1 \]