396 Investigate the fixed-point equation

$$strange = \S n: nat\cdot \forall m: strange\cdot \neg\ m+1: n\times nat$$

After trying the question, scroll down to the solution.

§ Clearly $strange: nat$. For any $n: nat$,

$$\begin{aligned} n: strange \;&=\; \forall m: strange\cdot \neg\, m+1: n{\times}nat \\ &=\; (\,strange+1 \text{ and } n{\times}nat \text{ are disjoint}) \end{aligned}$$

Let's see if $0: strange$.

$$\begin{aligned} & 0: strange \\ = \;& \forall m: strange\cdot \neg\, m+1: 0{\times}nat && \text{Increase domain and multiply} \\ \Leftarrow \;& \forall m: nat\cdot \neg\, m+1: 0 \\ = \;& \top \end{aligned}$$

So we see 0 is in. Let's see about 1 .

$$\begin{aligned} & 1: strange \\ = \;& \forall m: strange\cdot \neg\, m+1: 1{\times}nat && \text{Specialize } m \text{ to } 0 \\ \Rightarrow \;& \neg\, 0+1: nat \\ = \;& \bot \end{aligned}$$

So we see that 1 is out. Let's try 2 .

$$\begin{aligned} & 2: strange \\ = \;& \forall m: strange\cdot \neg\, m+1: 2{\times}nat \\ = \;& strange:\ 2{\times}nat \end{aligned}$$

We see that 2 is in if and only if they are all even. We similarly don't find out whether any larger numbers are in or out. Let's try recursive construction.

$$\begin{aligned} strange_0 &= null \\ strange_1 &= \S n: nat\cdot \forall m: null\cdot \neg\, m+1: n{\times}nat \\ &= \S n: nat\cdot \top \\ &= nat \\ strange_2 &= \S n: nat\cdot \forall m: nat\cdot \neg\, m+1: n{\times}nat \\ &= 0 \\ strange_3 &= \S n: nat\cdot \forall m: 0\cdot \neg\, m+1: n{\times}nat \\ &= \S n: nat\cdot \neg\, 0+1: n{\times}nat \\ &= 0,\, 2+nat \\ strange_4 &= \S n: nat\cdot \forall m: 0,\, 2+nat\cdot \neg\, m+1: n{\times}nat \\ &= \S n: nat\cdot (\forall m: 0\cdot \neg\, m+1: n{\times}nat) \wedge (\forall m: 2+nat\cdot \neg\, m+1: n{\times}nat) \\ &= 0,\, null \\ &= strange_2 \end{aligned}$$

We see that from $strange_1$ on, 0 stays in; from $strange_2$ on, 1 stays out; all other numbers are alternately in and out. Recursive construction tells us what we already knew, and no more. However, for any prime number p , the multiples of that prime $p{\times}nat$ form a fixed-point of the $strange$ equation.