390 Let $A \mathbin{\bar{,}} B$ be bunch removal: from bunch A remove any elements in bunch B . The operator $\bar{,}$ has precedence level 4, and is defined by the axiom

$$x{:}\ A \mathbin{\bar{,}} B \ = \ x{:}\ A \ \land \ \lnot\ x{:}\ B$$

For each of the following fixed-point equations, what does recursive construction yield? Does it satisfy the fixed-point equation?

(a) $Q \ = \ nat \mathbin{\bar{,}} (Q+3)$

(b) $D \ = \ 0, (D+1) \mathbin{\bar{,}} (D-1)$

(c) $E \ = \ nat \mathbin{\bar{,}} (E+1)$

(d) $F \ = \ 0, (nat \mathbin{\bar{,}} F)+1$

After trying the question, scroll down to the solution.

(a) $Q \;=\; nat \mathbin{\neg} (Q+3)$

§ $Q_0 \;=\; null$

$Q_1 \;=\; nat \mathbin{\neg} (null+3) \;=\; nat \mathbin{\neg} null \;=\; nat$

$Q_2 \;=\; nat \mathbin{\neg} (nat+3) \;=\; 0, 1, 2$

$Q_3 \;=\; nat \mathbin{\neg} ((0, 1, 2)+3) \;=\; nat \mathbin{\neg} (3, 4, 5) \;=\; 0, 1, 2, nat+6$

$Q_4 \;=\; nat \mathbin{\neg} ((0, 1, 2, nat+6)+3) \;=\; nat \mathbin{\neg} (3, 4, 5, nat+9) \;=\; 0, 1, 2, 6, 7, 8$

$Q_5 \;=\; nat \mathbin{\neg} ((0, 1, 2, 6, 7, 8)+3) \;=\; nat \mathbin{\neg} (3, 4, 5, 9, 10, 11)$

$\;=\; 0, 1, 2, 6, 7, 8, nat+12$

Time for a guess. It looks like there are two patterns: the even index pattern and the odd index pattern. So I guess

$Q_{2\times n} \;=\; 6\times(0,..n) + (0,..3)$

$Q_{2\times n+1} \;=\; 6\times(0,..n) + (0,..3),\; (6\times n,..\infty)$

From the even case, I propose

$Q_\infty \;=\; 6\times nat + (0,..3)$

and now I have to check whether it satisfies the equation. Starting with the right side,

$nat \mathbin{\neg} (Q_\infty+3)$

$=\; nat \mathbin{\neg} (6\times nat + (0,..3) + 3)$

$=\; nat \mathbin{\neg} (6\times nat + (3,..6))$ here is an informal expansion

$=\; nat \mathbin{\neg} ((0, 6, 12, 18, 24, ...) + (3,..6))$ and an informal addition

$=\; nat \mathbin{\neg} (3, 4, 5, 9, 10, 11, 15, 16, 17, 21, 22, 23, 27, 28, 29, ...)$

$=\; 0, 1, 2, 6, 7, 8, 12, 13, 14, 18, 19, 20, 24, 25, 26, ...$

$=\; (0, 6, 12, 18, 24, ...) + (0,..3)$

$=\; 6\times nat + (0,..3)$

$=\; Q_\infty$

So it does satisfy the equation. From the odd case, I propose

$Q_\infty \;=\; 6\times nat + (0,..3),\; \infty$

and now I have to check whether it satisfies the equation. Starting with the right side,

$nat \mathbin{\neg} (Q_\infty+3)$

$=\; nat \mathbin{\neg} ((6\times nat + (0,..3), \infty) + 3)$

$=\; nat \mathbin{\neg} (6\times nat + (3,..6), \infty)$ $\neg(\infty\!: nat)$

$=\; nat \mathbin{\neg} (6\times nat + (3,..6))$ as before

$=\; 6\times nat + (0,..3)$

$\ne\; 6\times nat + (0,..3),\; \infty$

$=\; Q_\infty$

so this is not a solution.

(b) $D \;=\; 0, (D+1) \mathbin{\neg} (D-1)$

§ $D_0 \;=\; null$

$D_1 \;=\; 0, (D_0+1) \mathbin{\neg} (D_0-1)$

$\;=\; 0, (null+1) \mathbin{\neg} (null-1)$

$\;=\; 0, null \mathbin{\neg} null$

$\;=\; 0$

$D_2 \;=\; 0, (D_1+1) \mathbin{\neg} (D_1-1)$

$\;=\; 0, (0+1) \mathbin{\neg} (0-1)$

$\;=\; 0, 1 \mathbin{\neg} -1$

$\;=\; 0, 1$

$D_3 \;=\; 0, (D_2+1) \mathbin{\neg} (D_2-1)$

$\;=\; 0, ((0, 1)+1) \mathbin{\neg} ((0, 1)-1)$

$\;=\; 0, (1, 2) \mathbin{\neg} (-1, 0)$

$\;=\; 0, 1, 2$

$D_4 \;=\; 0, (D_3+1) \mathbin{\neg} (D_3-1)$

$\;=\; 0, ((0, 1, 2)+1) \mathbin{\neg} ((0, 1, 2)-1)$

$$
\begin{aligned}
&= 0, (1, 2, 3)\bar{,}(-1, 0, 1)\\
&= 0, 2, 3\\
D_5 &= 0, (D_4+1)\bar{,}(D_4-1)\\
&= 0, ((0, 2, 3)+1)\bar{,}((0, 2, 3)-1)\\
&= 0, (1, 3, 4)\bar{,}(-1, 1, 2)\\
&= 0, 3, 4\\
D_6 &= 0, (D_5+1)\bar{,}(D_5-1)\\
&= 0, ((0, 3, 4)+1)\bar{,}((0, 3, 4)-1)\\
&= 0, (1, 4, 5)\bar{,}(-1, 2, 3)\\
&= 0, 1, 4, 5\\
D_7 &= 0, (D_6+1)\bar{,}(D_6-1)\\
&= 0, ((0, 1, 4, 5)+1)\bar{,}((0, 1, 4, 5)-1)\\
&= 0, (1, 2, 5, 6)\bar{,}(-1, 0, 3, 4)\\
&= 0, 1, 2, 5, 6\\
D_8 &= 0, (D_7+1)\bar{,}(D_7-1)\\
&= 0, ((0, 1, 2, 5, 6)+1)\bar{,}((0, 1, 2, 5, 6)-1)\\
&= 0, (1, 2, 3, 6, 7)\bar{,}(-1, 0, 1, 4, 5)\\
&= 0, 2, 3, 6, 7
\end{aligned}
$$

It's still hard to see the patterns, so maybe we have to go a bit farther. Then we see

$$
\begin{aligned}
D_{4\times n+1} &= 0, 4\times(0,..n) + (3, 4)\\
D_{4\times n+2} &= 0, 1, 4\times(0,..n) + (4, 5)\\
D_{4\times n+3} &= 0, 1, 2, 4\times(0,..n) + (5, 6)\\
D_{4\times n+4} &= 0, 2, 3, 4\times(0,..n) + (6, 7)
\end{aligned}
$$

We have a choice of four possible answers for D_∞ , but none of them satisfies the equation. Recursive construction fails.

(c) $E = nat\bar{,}(E+1)$

§

$$
\begin{aligned}
E_0 &= null\\
E_1 &= nat\\
E_2 &= 0\\
E_3 &= 0, nat+2\\
E_4 &= 0, 2\\
E_5 &= 0, 2, nat+4\\
E_{2\times n} &= 2\times(0,..n)\\
E_{2\times n+1} &= 2\times(0,..n), nat+2\times n
\end{aligned}
$$

From the even case, we propose

$E_\infty = 2\times nat$

which satisfies the equation. From the odd case, we propose

$E_\infty = 2\times nat, \infty$

which does not satisfy the equation.

(d) $F = 0, (nat\bar{,}F)+1$

§

$$
\begin{aligned}
F_0 &= null\\
F_1 &= nat\\
F_2 &= 0\\
F_3 &= 0, nat+2\\
F_4 &= 0, 2\\
F_5 &= 0, 2, nat+4\\
F_{2\times n} &= 2\times(0,..n)\\
F_{2\times n+1} &= 2\times(0,..n), nat+2\times n
\end{aligned}
$$

From the even case, we propose

$$F_\infty = 2{\times}nat$$

which satisfies the fixed-point equation. From the odd case, we propose

$$F_\infty = 2{\times}nat, \infty$$

which does not satisfy the fixed-point equation.