Let $\cdot$ be a two-operand infix operator (let's give it precedence 3) whose operands and result are of some type $T$. Let $\diamond$ be a two-operand infix operator (let's give it precedence 7) whose operands are of type $T$ and whose result is binary, defined by the axiom

$$a \diamond b = a \cdot b = a$$

(a) Prove if $\cdot$ is idempotent then $\diamond$ is reflexive.

§

\[
a \cdot a = a
\]

use axiom

\[
a \cdot a = a
\]

(b) Prove if $\cdot$ is associative then $\diamond$ is transitive.

§

\[
a \diamond b \land b \diamond c
\]

use axiom 2 times

\[
a \cdot b = a \land b \cdot c = b
\]

idempotence of $\land$

\[
a \cdot b = a \land a \cdot b = a \land b \cdot c = b
\]

use third conjunct to replace $b$ in second

\[
a \cdot b = a \land a \cdot (b \cdot c) = a \land b \cdot c = b
\]

specialize: drop third conjunct

\[
a \cdot b = a \land a \cdot c = a
\]

use associativity

\[
a \cdot b = a \land (a \cdot b) \cdot c = a
\]

use first conjunct to replace $a \cdot b$ in second

\[
a \cdot b = a \land a \cdot c = a
\]

specialize: drop first conjunct

\[
a \cdot c = a
\]

use axiom

\[
a \diamond c
\]

(c) Prove if $\cdot$ is symmetric then $\diamond$ is antisymmetric.

§

\[
a \diamond b \land b \diamond a
\]

use axiom 2 times

\[
a \cdot b = a \land b \cdot a = b
\]

use symmetry of $=$ and $\cdot$

\[
a = a \cdot b \land a \cdot b = b
\]

transitivity of $=$

\[
a \Rightarrow a = b
\]

(d) If $T$ is the binary values and $\cdot$ is $\land$, what is $\diamond$?

§

\[
a \Rightarrow b = a \land b = a \text{ so } \diamond \text{ is } \Rightarrow.
\]

(e) If $T$ is the binary values and $\cdot$ is $\lor$, what is $\diamond$?

§

\[
a \Leftarrow b = a \lor b = a \text{ so } \diamond \text{ is } \Leftarrow.
\]

(f) If $T$ is the natural numbers and $\diamond$ is $\leq$, what is $\cdot$?

§

\[
a \leq b = \min a b = a \text{ so } \cdot \text{ is } \min.
\]

(g) The axiom defines $\diamond$ in terms of $\cdot$. Can it be inverted, so that $\cdot$ is defined in terms of $\diamond$?

§

If $T$ is the binary values we can invert as follows: $a \cdot b = a \diamond b = a$. If $T$ is anything else, we can invert under the assumption $a \diamond b \lor b \diamond a$. The inversion is

\[
a \cdot b = \text{if } a \diamond b \text{ then } a \text{ else } b
\]