Let \( A \setminus B \) be the difference between bunch \( A \) and bunch \( B \). The operator \( \setminus \) has precedence level 4, and is defined by the axiom

\[ x : A \setminus B \equiv x : A \land \neg x : B \]

For each of the following fixed-point equations, what does recursive construction yield? Does it satisfy the fixed-point equation?

(a) \( Q = \text{nat} \setminus (Q+3) \)

\[ Q_0 = \text{null} \]

\[ Q_1 = \text{nat} \setminus (\text{null}+3) = \text{nat} \setminus \text{null} = \text{nat} \]

\[ Q_2 = \text{nat} \setminus (\text{nat}+3) = 0, 1, 2 \]

\[ Q_3 = \text{nat} \setminus ((0, 1, 2)+3) = \text{nat} \setminus (3, 4, 5) = 0, 1, 2, \text{nat}+6 \]

\[ Q_4 = \text{nat} \setminus ((0, 1, 2, \text{nat}+6)+3) = \text{nat} \setminus (3, 4, 5, \text{nat}+9) = 0, 1, 2, 6, 7, 8 \]

\[ Q_5 = \text{nat} \setminus ((0, 1, 2, 6, 7, 8)+3) = \text{nat} \setminus (3, 4, 5, 9, 10, 11) \]

\[ = 0, 1, 2, 6, 7, 8, \text{nat}+12 \]

Time for a guess. It looks like there are two patterns: the even index pattern and the odd index pattern. So I guess

\[ Q_{2n} = 6n(0..n) + (0..3) \]

\[ Q_{2n+1} = 6n(0..n) + (0..3), (6n,..\infty) \]

From the even case, I propose

\[ Q_{\infty} = 6n\text{nat} + (0..3) \]

and now I have to check whether it satisfies the equation. Starting with the right side,

\[ \text{nat} \setminus (Q_{\infty}+3) \]

\[ = \text{nat} \setminus (6n\text{nat} + (0..3) + 3) \]

\[ = \text{nat} \setminus (6n\text{nat} + (3..6)) \]

here is an informal expansion

\[ = \text{nat} \setminus ((0, 6, 12, 18, 24, ...) + (3..6)) \]

and an informal addition

\[ = \text{nat} \setminus (3, 4, 5, 9, 10, 11, 15, 16, 17, 21, 22, 23, 27, 28, 29, ...) \]

\[ = 0, 1, 2, 6, 7, 8, 12, 13, 14, 18, 19, 20, 24, 25, 26, ... \]

\[ = (0, 6, 12, 18, 24, ...) + (0..3) \]

\[ = 6n\text{nat} + (0..3) \]

\[ = Q_{\infty} \]

So it does satisfy the equation. From the odd case, we can't make a proposal because we can't simplify \( \infty,..\infty \).

(b) \( D = 0, (D+1) \setminus (D–1) \)

\[ D_0 = \text{null} \]

\[ D_1 = 0, (D_0 +1) \setminus (D_0–1) \]

\[ = 0, (\text{null} +1) \setminus (\text{null}–1) \]

\[ = 0, \text{null} \setminus \text{null} \]

\[ = 0 \]

\[ D_2 = 0, (D_1 +1) \setminus (D_1–1) \]

\[ = 0, (0+1) \setminus (0–1) \]

\[ = 0, 1 \setminus 1 \]

\[ = 0, 1 \]

\[ D_3 = 0, (D_2 +1) \setminus (D_2–1) \]

\[ = 0, ((0, 1)+1) \setminus ((0, 1)–1) \]

\[ = 0, (1, 2)\setminus(–1, 0) \]

\[ = 0, 1, 2 \]

\[ D_4 = 0, (D_3 +1) \setminus (D_3–1) \]

\[ = 0, ((0, 1, 2)+1) \setminus ((0, 1, 2)–1) \]

\[ = 0, (1, 2, 3)\setminus(–1, 0, 1) \]

\[ = 0, 2, 3 \]
\[
D_5 = 0, (D_4+1) \setminus (D_4-1) \\
= 0, ((0, 2, 3)+1) \setminus ((0, 2, 3)-1) \\
= 0, (1, 3, 4)\setminus(-1, 1, 2) \\
= 0, 3, 4
\]
\[
D_6 = 0, (D_5+1) \setminus (D_5-1) \\
= 0, ((0, 3, 4)+1) \setminus ((0, 3, 4)-1) \\
= 0, (1, 4, 5)\setminus(-1, 2, 3) \\
= 0, 1, 4, 5
\]
\[
D_7 = 0, (D_6+1) \setminus (D_6-1) \\
= 0, ((0, 1, 4, 5)+1) \setminus ((0, 1, 4, 5)-1) \\
= 0, (1, 2, 5, 6)\setminus(-1, 0, 3, 4) \\
= 0, 1, 2, 5, 6
\]
\[
D_8 = 0, (D_7+1) \setminus (D_7-1) \\
= 0, ((0, 1, 2, 5, 6)+1) \setminus ((0, 1, 2, 5, 6)-1) \\
= 0, (1, 2, 3, 6, 7)\setminus(-1, 0, 1, 4, 5) \\
= 0, 1, 2, 3, 6, 7
\]

It's still hard to see the patterns, so maybe we have to go a bit farther. Then we see
\[
D_{4\times n+1} = 0, 4\times(0..n) + (3, 4) \\
D_{4\times n+2} = 0, 1, 4\times(0..n) + (4, 5) \\
D_{4\times n+3} = 0, 1, 2, 4\times(0..n) + (5, 6) \\
D_{4\times n+4} = 0, 2, 3, 4\times(0..n) + (6, 7)
\]

We have a choice of four possible answers for \(D_\infty\), but none of them satisfies the equation. Recursive construction fails.

(c) \( E = \text{nat} \setminus (E+1) \)

\[
E_0 = \text{null} \\
E_1 = \text{nat} \\
E_2 = 0 \\
E_3 = 0, \text{nat}+2 \\
E_4 = 0, 2 \\
E_5 = 0, 2, \text{nat}+4 \\
E_{2\times n} = 2\times(0..n) \\
E_{2\times n+1} = 2\times(0..n), \text{nat}+2\times n
\]

From the even case, we propose
\( E_\infty = 2\times \text{nat} \)
which satisfies the equation. From the odd case, we propose
\( E_\infty = 2\times \text{nat}, \infty \)
which does not satisfy the equation.

(d) \( F = 0, (\text{nat} \setminus F)+1 \)

\[
F_0 = \text{null} \\
F_1 = \text{nat} \\
F_2 = 0 \\
F_3 = 0, \text{nat}+2 \\
F_4 = 0, 2 \\
F_5 = 0, 2, \text{nat}+4 \\
F_{2\times n} = 2\times(0..n) \\
F_{2\times n+1} = 2\times(0..n), \text{nat}+2\times n
\]

From the even case, we propose
\( F_\infty = 2\times \text{nat} \)
which satisfies the fixed-point equation. From the odd case, we propose
\( F_{\infty} = 2^{\times \text{nat}, \infty} \)

which does not satisfy the fixed-point equation.