Suppose we define while \( b \) do \( P \) od by ordinary construction and induction, ignoring time.

\[
\text{if } b \text{ then } P. \text{ while } b \text{ do } P \text{ od else ok fi } \Leftarrow \text{ while } b \text{ do } P \text{ od}
\]

\[
\forall \sigma, \sigma' . \text{ if } b \text{ then } P. W \text{ else ok fi } \Leftarrow W \Rightarrow \forall \sigma, \sigma' . \text{ while } b \text{ do } P \text{ od } \Leftarrow W
\]

Prove that fixed-point construction and induction

\[
\text{while } b \text{ do } P \text{ od } = \text{ if } b \text{ then } P. \text{ while } b \text{ do } P \text{ od else ok fi}
\]

\[
\forall \sigma, \sigma' . W = \text{ if } b \text{ then } P. W \text{ else ok fi } \Rightarrow \forall \sigma, \sigma' . \text{ while } b \text{ do } P \text{ od } \Leftarrow W
\]

are theorems.

§ Start with the ordinary construction axiom, universally quantified

\[
\forall \sigma, \sigma' . \text{ if } b \text{ then } P. \text{ while } b \text{ do } P \text{ od else ok fi } \Leftarrow \text{ while } b \text{ do } P \text{ od}
\]

by monotonicity of dependent composition (stepwise refinement)

\[
\Rightarrow \forall \sigma, \sigma' . (P. \text{ if } b \text{ then } P. \text{ while } b \text{ do } P \text{ od else ok fi} \Leftarrow P. \text{ while } b \text{ do } P)
\]

by monotonicity of if (stepwise refinement)

\[
\Rightarrow \forall \sigma, \sigma' . (\text{if } b \text{ then } P. \text{ if } b \text{ then } P. \text{ while } b \text{ do } P \text{ od else ok fi} \text{ else ok fi} \Leftarrow \text{if } b \text{ then } P. \text{ while } b \text{ do } P \text{ od else ok fi})
\]

This is the antecedent of the induction axiom with \( W \) replaced by \( \text{if } b \text{ then } P. \text{ while } b \text{ do } P \text{ od else ok fi} \), hence we conclude its consequent with the same replacement.

\[
\Rightarrow \forall \sigma, \sigma' . \text{ while } b \text{ do } P \text{ od } \Leftarrow \text{if } b \text{ then } P. \text{ while } b \text{ do } P \text{ od else ok fi}
\]

This is one half of fixed-point construction, and the ordinary construction axiom is the other half. That proves fixed-point construction. As for fixed-point induction, it is immediate from ordinary induction just by strengthening the antecedent. It is interesting to note that ordinary construction and induction can be stated together as one axiom just by strengthening the main connective in induction:

\[
\forall \sigma, \sigma' . \text{ if } b \text{ then } P. W \text{ else ok fi } \Leftarrow W \Rightarrow \forall \sigma, \sigma' . \text{ while } b \text{ do } P \text{ od } \Leftarrow W
\]