382 Function f is called monotonic if $\forall i, j: \square f \cdot\ i \le j \ \Rightarrow\ f\,i \le f\,j$.

(a) Prove f is monotonic if and only if $\forall i, j \cdot\ f\,i < f\,j \ \Rightarrow\ i < j$.

(b) Let $f: int \rightarrow int$. Prove f is monotonic if and only if $\forall i \cdot\ f\,i \le f\,(i+1)$.

(c) Let $f: nat \rightarrow nat$ be such that $\forall n \cdot f\,f\,n < f\,(n+1)$. Prove f is the identity function. Hints: First prove $\forall n \cdot\ n \le f\,n$. Then prove f is monotonic. Then prove $\forall n \cdot f\,n \le n$.

After trying the question, scroll down to the solution.

(a) Prove f is monotonic if and only if $f\,i < f\,j \;\Rightarrow\; i<j$.

§ $\quad \forall i,j\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j$ — contrapositive law

$=\quad \forall i,j\cdot\; \neg(f\,i \le f\,j) \;\Rightarrow\; \neg(i\le j)$ — a generic law

$=\quad \forall i,j\cdot\; f\,i > f\,j \;\Rightarrow i>j$ — generic mirror, twice

$=\quad \forall i,j\cdot\; f\,j < f\,i \;\Rightarrow j<i$ — rename i to j and j to i

$=\quad \forall j,i\cdot\; f\,i < f\,j \;\Rightarrow i<j$

(b) Let $f: int\to int$. Prove f is monotonic if and only if $f\,i \le f\,(i+1)$.

§ In one direction,

$\quad \forall i,j: int\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j$ — specialize j as $i+1$

$\Longrightarrow\quad \forall i: int\cdot\; i \le i+1 \;\Rightarrow\; f\,i \le f\,(i+1)$ — $i \le i+1$ is a theorem of arithmetic

$=\quad \forall i: int\cdot\; f\,i \le f\,(i+1)$

In the other direction,

$\quad (\forall i: int\cdot\; f\,i \le f\,(i+1)) \Rightarrow (\forall i,j: int\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j)$

$=\quad (\forall i: int\cdot\; f\,i \le f\,(i+1)) \Rightarrow (\forall i,j: nat, -nat\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j)$ — a basic quantifier law

$=\quad (\forall i: int\cdot\; f\,i \le f\,(i+1)) \;\Rightarrow\; (\forall i,j: nat\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j) \wedge (\forall i,j: -nat\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j)$ — a binary distributive law

$=\quad ((\forall i: int\cdot\; f\,i \le f\,(i+1)) \;\Rightarrow\; (\forall i,j: nat\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j))$
$\quad \wedge\; ((\forall i: int\cdot\; f\,i \le f\,(i+1)) \;\Rightarrow\; (\forall i,j: -nat\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j))$

I will prove the first (top) conjunct and the last (bottom) conjunct separately. To prove the first conjunct, I prove $\forall i,j: nat\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j$ with context $\forall i: int\cdot\; f\,i \le f\,(i+1)$. That means I prove $\forall i,j: nat\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j$ while assuming $\forall i: int\cdot\; f\,i \le f\,(i+1)$ as a local axiom.

$\quad \forall i,j: nat\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j$ — write the quantifiers separately

$=\quad \forall i: nat\cdot \forall j: nat\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j$ — nat induction on j

$\Longleftarrow\quad \forall i: nat\cdot\; (i\le 0 \;\Rightarrow\; f\,i \le f\,0)$ — For $i: nat$, $(i\le 0)=(i=0)$, and $f\,0 \le f\,0$ is $\top$
$\quad \wedge\; (\forall j: nat\cdot (i\le j \;\Rightarrow\; f\,i \le f\,j) \Rightarrow (i \le j+1 \;\Rightarrow\; f\,i \le f\,(j+1)))$

$=\quad \forall i,j: nat\cdot\; (i\le j \;\Rightarrow\; f\,i \le f\,j) \Rightarrow (i \le j+1 \;\Rightarrow\; f\,i \le f\,(j+1))$ — generic inclusive law

$=\quad \forall i,j: nat\cdot\; (i\le j \;\Rightarrow\; f\,i \le f\,j) \Rightarrow (i = j+1 \vee i < j+1 \;\Rightarrow\; f\,i \le f\,(j+1))$ — antidistributive law

$=\quad \forall i,j: nat\cdot\; (i\le j \;\Rightarrow\; f\,i \le f\,j)$
$\quad \Rightarrow ((i = j+1 \;\Rightarrow\; f\,i \le f\,(j+1)) \wedge (i < j+1 \;\Rightarrow\; f\,i \le f\,(j+1)))$
If $i = j+1$ then $f\,i \le f\,(j+1)$ by generic reflexive law. Also $(i < j+1)=(i\le j)$.

$=\quad \forall i,j: nat\cdot\; (i\le j \;\Rightarrow\; f\,i \le f\,j) \Rightarrow (i\le j \;\Rightarrow\; f\,i \le f\,(j+1))$ — portation

$=\quad \forall i,j: nat\cdot\; (i\le j \;\Rightarrow\; f\,i \le f\,j) \wedge i\le j \;\Rightarrow\; f\,i \le f\,(j+1)$ — discharge

$=\quad \forall i,j: nat\cdot\; i\le j \;\wedge\; f\,i \le f\,j \;\Rightarrow\; f\,i \le f\,(j+1)$ — add context with i changed to j

$=\quad \forall i,j: nat\cdot\; i\le j \;\wedge\; f\,i \le f\,j \;\wedge\; f\,j \le f\,(j+1) \;\Rightarrow\; f\,i \le f\,(j+1)$ — transitivity

$\Longleftarrow\quad \forall i,j: nat\cdot\; i\le j \;\wedge\; f\,i \le f\,(j+1) \;\Rightarrow\; f\,i \le f\,(j+1)$ — reflexivity

$\Longleftarrow\quad \top$

To prove the last (bottom) conjunct, I prove $\forall i,j: -nat\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j$ with context $\forall i: int\cdot\; f\,i \le f\,(i+1)$.

$\quad \forall i,j: -nat\cdot\; i\le j \;\Rightarrow\; f\,i \le f\,j$ — change of variables

$=\quad \forall i,j: nat\cdot\; -i\le -j \;\Rightarrow\; f\,(-i) \le f\,(-j)$ — write the quantifiers separately

$=\quad \forall i: nat\cdot \forall j: nat\cdot\; -i\le -j \;\Rightarrow\; f\,(-i) \le f\,(-j)$ — nat induction on j

$\Longleftarrow\quad \forall i: nat\cdot\; (-i\le 0 \;\Rightarrow\; f\,(-i) \le f\,0)$
$\quad \wedge\; (\forall j: nat\cdot (-i\le -j \;\Rightarrow\; f\,(-i) \le f\,(-j)) \Rightarrow (-i\le -(j+1) \;\Rightarrow\; f\,(-i) \le f\,(-(j+1))))$

$\cdots\quad$ UNFINISHED

$\Longleftarrow\quad \top$

(c) Let $f: nat\to nat$ be such that $\forall n\cdot f\,f\,n < f\,(n+1)$. Prove f is the identity function. Hints: First prove $\forall n\cdot\; n \le f\,n$. Then prove f is monotonic. Then prove $\forall n\cdot f\,n \le n$.

§ We first prove $\forall n\cdot\ n \le f\,n$ by induction on n .

Base case: $n{=}0$: $0 \le f\,0$ because f: $nat \to nat$

Assume $\forall n\cdot\ i{\le}n \Rightarrow i \le f\,n$ as induction hypothesis. We must now prove

$\forall n\cdot\ i{+}1{\le}n \Rightarrow i{+}1 \le f\,n$ which we prove by induction on n .

Base case: $n{=}0$: $i{+}1 \le 0 \Rightarrow i{+}1 \le f\,0$ has a false antecedent.

Assume $n \le f\,n$ as induction hypothesis. We must now prove

$n{+}1 \le f\,(n{+}1)$

$=$ $n < f\,(n{+}1)$ stick two terms in between

$\Leftarrow$ $n \le f\,n \le f\,f\,n < f\,(n{+}1)$

Use the induction hypothesis for $n \le f\,n$. Use it again for $f\,n \le f\,f\,n$ with n instantiated as $f\,n$. Use the given information $f\,f\,n < f\,(n{+}1)$ for the final piece. Now we have proven $\forall n\cdot\ n \le f\,n$, which means that f lies on or above the diagonal. Next we prove $\forall n\cdot f\,n < f\,(n{+}1)$.

$f\,n$ use $n \le f\,n$ with n instantiated as $f\,n$.

$\le$ $f\,f\,n$ use the given information $f\,f\,n < f\,(n{+}1)$

$<$ $f\,(n{+}1)$

Now we prove $\forall n\cdot f\,n \le n$.

$f\,n \le n$

$=$ $f\,n < n{+}1$ use part (a) with $f\,n$ as i and $n{+}1$ as j

$\Leftarrow$ $f\,f\,n < f\,(n{+}1)$ use the given information.

$=$ $\top$

THIS PROOF NEEDS TO BE MADE FULLY CALCULATIONAL.