Prove the law $\text{nat} = 0..\infty$ from the other laws.

After trying the question, scroll down to a solution attempt.
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I'll start by proving half of it.

\( \text{nat} : 0,..\infty \)

In this half, I am trying to prove \( \text{nat} \) isn't too big, so I'll need \( \text{nat} \) induction

\( 0, B+1 : B \Rightarrow \text{nat} : B \)

Here we go:

\[
\begin{align*}
\Leftarrow & \quad 0, (0,..\infty)+1 : 0,..\infty \\
\Rightarrow & \quad 0,0+1,..\infty+1 : 0,..\infty \\
\Rightarrow & \quad 0, 1,..\infty : 0,..\infty \\
\Rightarrow & \quad 0,..\infty : 0,..\infty \\
\Rightarrow & \quad \top
\end{align*}
\]

The other half is

\( 0,..\infty : \text{nat} \)

In this half, I am trying to prove that all of \( 0,..\infty \) is in \( \text{nat} \), so I should need \( \text{nat} \) construction.

\( 0, \text{nat}+1 : \text{nat} \)

So maybe I should prove

\( 0,..\infty : 0, \text{nat}+1 \)

and then the result will follow by transitivity. Also, these laws look like they might be useful:

\[
\begin{align*}
x, y: \text{xint} \land x\leq y & \Rightarrow (i: x,..y = i: \text{xint} \land x\leq i\leq y) \quad & \text{interval law} \\
A: B &= \forall x: A \cdot x; B \quad & \text{inclusion law} \\
V: W &= \forall v: V \cdot \exists w: W \cdot v=w \quad & \text{bunch-element conversion law} \\
A: B \land B; C & \Rightarrow A: C \quad & \text{transitivity}
\end{align*}
\]

But I am unable to find a proof.

Here's an attempt to prove both halves together.

\[
\begin{align*}
\top &= \forall i: \text{xint} \cdot \top & \text{identity} \\
&= \forall i: \text{xint} \cdot x, y: \text{xint} \land x\leq y \Rightarrow (i: x,..y = i: \text{xint} \land x\leq i\leq y) & \text{interval law} \\
&= \forall i: \text{xint} \cdot 0,..\infty = i: \text{xint} \land 0\leq i<\infty & \text{specialize} \\
&= \forall i: \text{xint} \cdot (i: 0,..\infty = i: \text{xint} \land 0\leq i<\infty) & \text{context provided by quantification} \\
&= \forall i: \text{xint} \cdot (i: 0,..\infty = \top \land 0\leq i<\infty) & \text{(quantifier applies to a function; function variable introduction is axiom in body)} \\
&= \forall i: \text{xint} \cdot 0,..\infty = \text{nat} & \text{identity} \\
&= \forall i: \text{xint} \cdot (i: 0,..\infty = 0\leq i<\infty) & \text{I can't justify this step} \\
&= \forall i: \text{xint} \cdot (i: 0,..\infty = i: \text{nat}) & \text{I can't justify this step either} \\
&= \forall i: \text{xint} \cdot 0,..\infty = \text{nat} & \text{idempotent} \\
&= 0,..\infty = \text{nat}
\end{align*}
\]

You might just say it's obvious that \( \text{nat} = 0,..\infty \), so why do we have to prove it? We have an application for our math: formal methods of software development. For that application, we want some binary expressions to be theorems (for example, \( \text{nat} = 0,..\infty \)), and we want other binary expressions to be antitheorems (for example, \( \text{nat} \neq 0,..\infty \)), and there are other binary expressions that we don't care about (for example, \( 0/0 = 1 \)). If we say \( \text{nat} = 0,..\infty \) is obvious, that just means it's obvious that we want it to be a theorem. If it cannot be proven, we need to add it to the theory. I have added it, but I think it was already there.