Can the law $\text{nat} = 0,\ldots,\infty$ be proven from the other laws?

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One half of it

$\text{nat}: 0,\ldots,\infty$

can be proven. In this half, I am trying to prove $\text{nat}$ isn't too big, so I'll need $\text{nat}$ induction

$0, B+1: B \implies \text{nat}: B$

Here we go:

$\text{nat}: 0,\ldots,\infty$

use induction with $0,\ldots,\infty$ for $B$

$\iff 0, (0,\ldots,\infty)+1: 0,\ldots,\infty$

addition distributes over bunch union

$\equiv 0, 0+1,\ldots,\infty+1: 0,\ldots,\infty$

arithmetic; $\infty$ absorbs additions

$\equiv 0, 0,\ldots,\infty$

reflexive

The other half is

$0,\ldots,\infty: \text{nat}$

In this half, I am trying to prove that all of $0,\ldots,\infty$ is in $\text{nat}$, so I should need $\text{nat}$ construction.

$0, \text{nat}+1: \text{nat}$

Also, these laws look like they might be useful:

$x, y: \text{xint} \land x \leq y \Rightarrow (i: x, y = i: \text{xint} \land x \leq i < y)$ interval law

$A: B \equiv \forall x: A \cdot x: B$ inclusion law

$V: W = \forall v: V \cdot \exists w: W \cdot v=w$ bunch-element conversion law

$A: B \land B: C \Rightarrow A: C$ transitivity

But I am unable to find a proof.

Here’s an attempt to prove both halves together.

$\top$ identity

$\equiv \forall i: \text{xint} \cdot \top$ interval law

$\equiv \forall i: \text{xint} \cdot x, y: \text{xint} \land x \leq y \Rightarrow (i: x, y = i: \text{xint} \land x \leq i < y)$ specialize

$\implies \forall i: \text{xint} \cdot 0,\ldots,\infty: \text{xint} \land 0 \leq i < \infty \Rightarrow (i: 0,\ldots,\infty = i: \text{xint} \land 0 \leq i < \infty)$ antecedent all $\top$

$\equiv \forall i: \text{xint} \cdot (i: 0,\ldots,\infty = i: \text{xint} \land 0 \leq i < \infty)$ context provided by quantification

(queueifier applies to a function; function variable introduction is axiom in body)

$\equiv \forall i: \text{xint} \cdot (i: 0,\ldots,\infty = \top \land 0 \leq i < \infty)$ identity

$\equiv \forall i: \text{xint} \cdot (i: 0,\ldots,\infty = 0 \leq i < \infty)$ I can't justify this step

$\equiv \forall i: \text{xint} \cdot (i: 0,\ldots,\infty = i: \text{nat})$ I can't justify this step either

$\equiv \forall i: \text{xint} \cdot 0,\ldots,\infty = \text{nat}$ idempotent

$\equiv 0,\ldots,\infty = \text{nat}$

You might just say it's obvious that $\text{nat} = 0,\ldots,\infty$, so why do we have to prove it? We have an application for our math: formal methods of software development. For that application, we want some binary expressions to be theorems (for example, $\text{nat} = 0,\ldots,\infty$), and we want other binary expressions to be antitheorems (for example, $\text{nat} \neq 0,\ldots,\infty$), and there are other binary expressions that we don't care about (for example, $0/0 = 1$). If we say $\text{nat} = 0,\ldots,\infty$ is obvious, that just means it's obvious that we want it to be a theorem. If it cannot be proven, we need to add it to the theory. I have added it, but I wonder if I had to, or if it was already there.