Can the law $\text{nat} = 0,..,\infty$ be proven from the other laws?

One half of it

$\text{nat} = 0,..,\infty$

can be proven. In this half, I am trying to prove $\text{nat}$ isn't too big, so I'll need $\text{nat}$ induction

$$0, B+1 : B \Rightarrow \text{nat} : B$$

Here we go:

- $\text{nat} = 0,..,\infty$
  - use induction with $0,..,\infty$ for $B$
- $0, (0,..,\infty)+1 : 0,..,\infty$
  - addition distributes over bunch union
- $0, 0+1,..,\infty+1 : 0,..,\infty$
  - arithmetic; $\infty$ absorbs additions
- $0, 1,..,\infty : 0,..,\infty$
  - reflexive

The other half is

$0,..,\infty : \text{nat}$

In this half, I am trying to prove that all of $0,..,\infty$ is in $\text{nat}$, so I should need $\text{nat}$ construction.

$$0, \text{nat}+1 : \text{nat}$$

Also, these laws look like they might be useful:

- $x : \text{int} \land y : \text{int} \land x \leq y \Rightarrow (i : x,..,y = i : \text{int} \land x \leq i < y)$
  - unnamed bunch law
- $A : B = \forall x : A, x : B$
  - inclusion law
- $V : W = \forall v : V, \exists w : W, v = w$
  - bunch-element conversion law
- $A : B \land B : C \Rightarrow A : C$
  - transitivity

But I am unable to find a proof.

Here's an attempt to prove both halves together.

$$\top$$
- identity
- interval bunch law
- $\forall i : \text{int} \land \top$
- specialize
- $\Rightarrow \forall i : \text{int} \land 0,..,\infty : \text{int} \land 0 \leq i < \infty \Rightarrow (i : 0,..,\infty = i : \text{int} \land 0 \leq i < \infty)$
  - I can't justify this step
- $\Rightarrow \forall i : \text{int} \land (i : 0,..,\infty = i : \text{int} \land 0 \leq i < \infty)$
  - I can't justify this step either
- $\Rightarrow \forall i : \text{int} \land 0,..,\infty = \text{nat}$
  - idempotent
- $= 0,..,\infty = \text{nat}$

I am wondering if we need to use the limit law

$$(\notin n \cdot n) = \infty$$

You might just say it's obvious that $\text{nat} = 0,..,\infty$, so why do we have to prove it? We have an application for our math: formal methods of software development. For that application, we want some binary expressions to be theorems (for example, $\text{nat} = 0,..,\infty$), and we want other binary expressions to be antitheorems (for example, $\text{nat} \neq 0,..,\infty$), and there are other binary expressions that we don't care about (for example, $0/0 = 1$). If we say $\text{nat} = 0,..,\infty$ is obvious, that just means it's obvious that we want it to be a theorem. If it cannot be proven, we need to add it to the theory. I have added it, but I wonder if I had to, or if it was already there.