367 Prove $\forall n: nat\cdot P\ n \ = \ \forall n: nat\cdot \forall m: 0,..n\cdot P\ m$

After trying the question, scroll down to the solution.

§ First prove forward implication.

$$
\begin{array}{lll}
& \forall n\text{: } nat\cdot\ \forall m\text{: } 0,..n\cdot\ P\ m & \text{induction} \\
\Leftarrow & (\forall m\text{: } 0,..0\cdot\ P\ m)\ \wedge\ (\forall n\text{: } nat\cdot\ (\forall m\text{: } 0,..n\cdot\ P\ m) \Rightarrow (\forall m\text{: } 0,..n{+}1\cdot\ P\ m)) & \\
& & \textit{null}\ \text{domain; split the domain } 0,..n{+}1 \\
= & \top\ \wedge\ \forall n\text{: } nat\cdot\ (\forall m\text{: } 0,..n\cdot\ P\ m) \Rightarrow (\forall m\text{: } 0,..n\cdot\ P\ m) \wedge (\forall m\text{: } n\cdot\ P\ m) & \\
& & \text{identity, a law of discharge, and } n \text{ is a one-point domain} \\
= & \forall n\text{: } nat\cdot\ (\forall m\text{: } 0,..n\cdot\ P\ m) \Rightarrow P\ n & \text{drop antecedent (or weaken it to } \top\text{)} \\
\Leftarrow & \forall n\text{: } nat\cdot\ P\ n &
\end{array}
$$

Now prove reverse implication.

$$
\begin{array}{lll}
& \forall n\text{: } nat\cdot\ \forall m\text{: } 0,..n\cdot\ P\ m & \text{use } \textit{nat} \text{ fixed-point construction} \\
= & \forall n\text{: } 0, nat{+}1\cdot\ \forall m\text{: } 0,..n\cdot\ P\ m & \text{a } \forall \text{ axiom} \\
= & (\forall n\text{: } 0\cdot\ \forall m\text{: } 0,..n\cdot\ P\ m) \wedge (\forall n\text{: } nat{+}1\cdot\ \forall m\text{: } 0,..n\cdot\ P\ m) & \text{specialize} \\
\Rightarrow & \forall n\text{: } nat{+}1\cdot\ \forall m\text{: } 0,..n\cdot\ P\ m & \text{change variable} \\
= & \forall k\text{: } nat\cdot\ \forall m\text{: } 0,..k{+}1\cdot\ P\ m & \text{specialize} \\
= & \forall k\text{: } nat\cdot\ P\ k & \text{rename} \\
= & \forall n\text{: } nat\cdot\ P\ n &
\end{array}
$$