366 Prove $\neg\ -1{:}\ nat$. Hint: You will need induction.

After trying the question, scroll down to the solution.

§ $\neg -1: nat$ — use law $nat = 0,..\infty$

$= \neg -1: 0,..\infty$ — definition of $,..$

$= \neg 0\le -1<\infty$ — drop conjunct

$\Leftarrow \neg 0\le -1$ — direction, translation, and various generic axioms

$= \top$

It seems we didn't need induction, but the law $nat = 0,..\infty$ needs induction to prove it. See Exercise 373.

Here is a predicate version. First, from either generalization or one-point we have

(0) $-1: nat \Rightarrow \exists n: nat\cdot -1=n$

Second, from connection (Galois)

$$n\le m \;=\; \forall k\cdot m\le k \Rightarrow n\le k$$

using contrapositive and specialization we have

(1) $n\le m \;\Longrightarrow\; -1<n \Rightarrow -1<m$

Now the proof

$\neg -1: nat$ — using contrapositive of (0)

$\Leftarrow \forall n: nat\cdot \neg -1=n$

$\Leftarrow \forall n: nat\cdot -1<n$ — now use induction

$\Leftarrow -1<0 \wedge (\forall n: nat\cdot -1<n \Rightarrow -1<n+1)$ — use (1) with $n+1$ for m

$\Leftarrow -1<0 \wedge (\forall n: nat\cdot n\le n+1)$ — translation twice

$= 0<1 \wedge (\forall n: nat\cdot 0\le 1)$

$= \top$