366 Prove \( \neg -1: \text{nat} \). Hint: You will need induction.

After trying the question, scroll down to the solution.
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\[ \neg -1 : \text{nat} \]

\[ \equiv \neg -1 : 0,\ldots,\infty \]

\[ \equiv \neg 0 \leq -1 < \infty \]

\[ \iff \neg 0 \leq -1 \]

\[ \equiv \top \]

It seems we didn't need induction, but the law \( \text{nat} = 0,\ldots,\infty \) needs induction to prove it. See Exercise 373.

Here is a predicate version. First, from either generalization or one-point we have

\[ (0) \quad -1 : \text{nat} \Rightarrow \exists n : \text{nat} -1 = n \]

Second, from connection (Galois)

\[ n \leq m = \forall k : m \leq k \Rightarrow n \leq k \]

using contrapositive and specialization we have

\[ (1) \quad n \leq m \Rightarrow -1 < n \Rightarrow -1 < m \]

Now the proof

\[ \neg -1 : \text{nat} \]

\[ \iff \forall n : \text{nat} \neg -1 = n \]

\[ \iff \forall n : \text{nat} \neg -1 < n \]

\[ \iff -1 < 0 \land (\forall n : \text{nat} -1 < n \Rightarrow -1 < n + 1) \]

\[ \iff -1 < 0 \land (\forall n : \text{nat} n \leq n + 1) \]

\[ \iff 0 < 1 \land (\forall n : \text{nat} 0 \leq 1) \]

\[ \equiv \top \]