354 (Monty Hall) Monty Hall is a game show host, and in this game there are three doors. A prize is hidden behind one of the doors. The contestant chooses a door. Monty then opens one of the doors, but not the door with the prize behind it, and not the door the contestant has chosen. Monty asks the contestant whether they (the contestant) would like to change their choice of door, or stay with their original choice. What should the contestant do?

After trying the question, scroll down to the solution.

§ Let p be the door where the prize is. Let c be the contestant's choice. Let m be the door Monty opens. If the contestant does not change their choice of door, the program, followed by the condition for winning, is:

$p:= rand\ 3.$ prize is hidden behind a door
$c:= rand\ 3.$ contestant chooses a door
$\mathbf{if}\ c=p$ If the contestant has chosen the door with the prize
$\mathbf{then\ if}\ rand\ 2\ \mathbf{then}\ m:= c\oplus 1\ \mathbf{else}\ m:= c\oplus 2\ \mathbf{fi}$ then Monty opens another door
$\mathbf{else}\ m:= 3-c-p\ \mathbf{fi}.$ otherwise Monty opens the remaining door
$ok.$ contestant decides not to switch
$c=p$ has the contestant won the prize?

The contestant has no idea where the prize is, so from the contestant's point of view, the prize is placed randomly. Then the contestant chooses a door at random. If the contestant happened to choose the door with the prize, then Monty chooses either one of the other two (where $\oplus$ is addition modulo 3); otherwise Monty must choose the one door that differs from both c and p . The next line ok is the contestant's decision not to change door. The final line $c=p$ is the question whether the contestant has won the prize. Now let's calculate. The assignments to m have no effect on c or p , and so they disappear. And ok is the identity for sequential composition.

$=$ $(p'\colon 0,..3)/3 \times (c'=c) \times (m'=m).$
$(c'\colon 0,..3)/3 \times (p'=p) \times (m'=m).$
$c=p$
$=$ $(p'\colon 0,..3)/3 \times (c'\colon 0,..3)/3 \times (m'=m).\ \ c=p$
$=$ $\Sigma p'', c'', m''\cdot\ (1/9) \times (m''=m) \times (c''=p'')$
$=$ $1/3$

Not surprisingly, the probability that the contestant wins is $1/3$. If the contestant takes the opportunity offered by Monty of switching their choice of door, the probability of winning is the remaining $2/3$. Just for fun, let's write the program, followed by the condition for winning, and calculate.

$p:= rand\ 3.$ prize is hidden behind a door
$c:= rand\ 3.$ contestant chooses a door
$\mathbf{if}\ c=p$ If the contestant has chosen the door with the prize
$\mathbf{then\ if}\ rand\ 2\ \mathbf{then}\ m:= c\oplus 1\ \mathbf{else}\ m:= c\oplus 2\ \mathbf{fi}$ then Monty opens another door
$\mathbf{else}\ m:= 3-c-p\ \mathbf{fi}.$ otherwise Monty opens the remaining door
$c:= 3-c-m.$ contestant decides to switch
$c=p$ has the contestant won the prize?

$=$ $(p'\colon 0,..3)/3 \times (c'=c) \times (m'=m).$
$(c'\colon 0,..3)/3 \times (p'=p) \times (m'=m).$
$\quad (c=p) \times (p'=p) \times (c'=c) \times ((m'=c\oplus 1)/2 + (m'=c\oplus 2)/2)$
$+\ \ (c\neq p) \times (p'=p) \times (c'=c) \times (m'=3-c-p)).$
$3-c-m = p$
$=$ $(p'\colon 0,..3)/3 \times (c'\colon 0,..3)/3 \times (m'=m).$
$\Sigma p'', c'', m''\cdot\ \ (\ \ (c=p) \times (p''=p) \times (c''=c) \times ((m''=c\oplus 1)/2 + (m''=c\oplus 2)/2)$
$\qquad +\ \ (c\neq p) \times (p''=p) \times (c''=c) \times (m''=3-c-p)\ )$
$\qquad \times\ \ (3-c''-m'' = p'')$
$=$ $(p'\colon 0,..3)/3 \times (c'\colon 0,..3)/3 \times (m'=m).$
$(c=p) \times ((c=p\oplus 1)/2 + (c=p\oplus 2)/2)\ \ +\ \ (c\neq p)$
$=$ $\Sigma p'', c'', m''\cdot\ (1/9) \times (m''=m) \times (c''\neq p'')$
$=$ $2/3$

The probability of winning is now $2/3$, so the contestant should switch.