The Fibonacci numbers $fib\ n$ are defined as follows.

\[
\begin{aligned}
&fib\ 0 = 0 \\
&fib\ 1 = 1 \\
&fib\ (n+2) = fib\ n + fib\ (n+1)
\end{aligned}
\]

Prove

I need a kind of induction that allows us to assume two previous cases as induction hypothesis in order to prove the next case. So I begin by proving the necessary induction law. All quantifications are over $nat$.

\[
\begin{aligned}
T \quad \text{predicate form of } nat \text{ induction} \\
= P0 \land (\forall n: Pn \Rightarrow P(n+1)) \Rightarrow (\forall n: Pn) \\
\Rightarrow Q0 \land Q1 \land (\forall n: \ Qn \land Q(n+1) \Rightarrow Q(n+1) \land Q(n+2)) \Rightarrow (\forall n: \ Qn \land Q(n+1))
\end{aligned}
\]

where $gcd$ is the greatest common divisor.

\[
\begin{aligned}
&fib\ (gcd\ n\ m) = gcd\ (fib\ n)\ (fib\ m) \\
\end{aligned}
\]

\[
\begin{aligned}
fib\ n \times fib\ (n+2) = (fib\ (n+1))^2 – (-1)^n
\end{aligned}
\]

\[
\begin{aligned}
fib\ (n+m+1) = fib\ n \times fib\ m + fib\ (n+1) \times fib\ (m+1)
\end{aligned}
\]

By induction on $n$. First, $n=0$.

\[
\begin{aligned}
fib\ n \times fib\ m + fib\ (n+1) \times fib\ (m+1) \\
= fib\ 0 \times fib\ m + fib\ 1 \times fib\ (m+1) \\
= 0 \times fib\ m + 1 \times fib\ (m+1) \\
= fib\ (0+m+1) \\
= fib\ (n+m+1)
\end{aligned}
\]

This induction needs a second base case: $n=1$.

\[
\begin{aligned}
fib\ n \times fib\ m + fib\ (n+1) \times fib\ (m+1) \\
= fib\ 1 \times fib\ m + fib\ 2 \times fib\ (m+1) \\
= 1 \times fib\ m + 1 \times fib\ (m+1) \\
= fib\ (m+2) \\
= fib\ (n+m+1)
\end{aligned}
\]

Now we are entitled to assume two previous cases:

\[
\begin{aligned}
fib\ (n+m-1) = fib\ (n-2) \times fib\ m + fib\ (n-1) \times fib\ (m+1) \\
fib\ (n+m) = fib\ (n-1) \times fib\ m + fib\ n \times fib\ (m+1)
\end{aligned}
\]

Add the left sides, and add the right sides.

\[
\begin{aligned}
fib\ (n+m-1) + fib\ (n+m) \\
= (fib\ (n-2) + fib\ (n-1)) \times fib\ m + (fib\ (n-1) + fib\ n) \times fib\ (m+1)
\end{aligned}
\]

Use the definition of $fib$ three times to obtain the desired result.

\[
\begin{aligned}
fib\ (n+m+2) = fib\ n \times fib\ (m+1) + fib\ (n+1) \times fib\ m + fib\ (n+1) \times fib\ (m+1)
\end{aligned}
\]

This proof is similar to part (c). First, $n=0$.

\[
\begin{aligned}
fib\ (m+2) = 0 \times fib\ (m+1) + 1 \times fib\ m + 1 \times fib\ (m+1)
\end{aligned}
\]

which follows from the definition of $fib$. Now the second base case: $n=1$.

\[
\begin{aligned}
fib\ (m+3) = 1 \times fib\ (m+1) + 1 \times fib\ m + 1 \times fib\ (m+1)
\end{aligned}
\]

add the last two terms

\[
\begin{aligned}
fib\ (m+3) = fib\ (m+1) + fib\ (m+2)
\end{aligned}
\]

which again follows. Now we are entitled to assume two previous cases:

\[
\begin{aligned}
fib\ (n+m) = fib\ (n-2) \times fib\ (m+1) + fib\ (n-1) \times fib\ m + fib\ (n-1) \times fib\ (m+1) \\
fib\ (n+m+1) = fib\ (n-1) \times fib\ (m+1) + fib\ n \times fib\ m + fib\ n \times fib\ (m+1)
\end{aligned}
\]

\[
\begin{aligned}
fib\ n \times fib\ (n+2) = (fib\ (n+1))^2 – (-1)^n
\end{aligned}
\]

\[
\begin{aligned}
fib\ (n+m+1) = fib\ n \times fib\ (m+1) + fib\ (n+1) \times fib\ m + fib\ (n+1) \times fib\ (m+1)
\end{aligned}
\]
Add the left sides, and add the right sides, using the definition on corresponding terms, to get the desired result.

(e) \[ \text{fib} (2n+1) = (\text{fib} n)^2 + (\text{fib} (n+1))^2 \]
§ This follows from part (c): just take \( m=n \).

(f) \[ \text{fib} (2n+2) = 2 \times \text{fib} n \times \text{fib} (n+1) + (\text{fib} (n+1))^2 \]
§ This follows from part (d): just take \( m=n \).

(g) \[ \forall n, k : \text{nat} \quad \text{fib} (kn) : \text{nat} \times \text{fib} n \]