A two-position switch is flipped some number of times. At each time (including initially, before the first flip) there is probability 1/2 of continuing to flip, and probability 1/2 of stopping. The probability that the switch ends in its initial state is 2/3, and the probability that it ends flipped is 1/3.

(a) Express the final state as a probability distribution.

§ $(ok + 1)/3$

When there is no state change, $ok$ is 1; when there is a state change, $ok$ is 0. Let the switch position be represented by binary variable $b$. Then I can rewrite it as

$((b' = b) + 1)/3$

I could further rewrite $b' = b$ as $2b' \times b - b' - b + 1$, but I see no point.

(b) Equate the distribution with a program describing the flips.

§ $(ok + 1)/3 \Leftrightarrow \text{if } 1/2 \text{ then } b := \neg b; \ (ok + 1)/3 \ \text{else } ok \ \text{fi}$

(c) Prove the equation.

§

\[
\begin{align*}
\text{if } 1/2 \text{ then } b := \neg b; \ (ok + 1)/3 \ \text{else } ok \ \text{fi} & \quad \text{replace first } ok \\
\equiv \ \text{if } 1/2 \text{ then } b := \neg b; \ ((b' = b) + 1)/3 \ \text{else } ok \ \text{fi} & \quad \text{use the Substitution Law} \\
\equiv \ \text{if } 1/2 \text{ then } ((b' = \neg b) + 1)/3 \ \text{else } ok \ \text{fi} & \quad (b' = \neg b) = \neg ok = 1 - ok \\
\equiv \ \text{if } 1/2 \text{ then } (2 - ok)/3 \ \text{else } ok \ \text{fi} & \quad \text{replace } if \\
\equiv \ (2 - ok)/3/2 + ok/2 & \quad \text{arithmetic} \\
\equiv \ (ok + 1)/3 &
\end{align*}
\]