Function $f$ is called monotonic if $i \leq j \Rightarrow f_i \leq f_j$.

(a) Prove $f$ is monotonic if and only if $f_i < f_j \Rightarrow i < j$.
§ Contrapositive, and switch $i$ and $j$.

(b) Let $f : \textrm{int} \to \textrm{int}$. Prove $f$ is monotonic if and only if $f_i \leq f(i+1)$.
§ In one direction, $i \leq j \Rightarrow f_i \leq f_j$ in instantiate $j$ as $i+1$.

In the other direction, we use Noetherian induction.

(c) Let $f : \textrm{nat} \to \textrm{nat}$ be such that $\forall n. ffn < f(n+1)$. Prove $f$ is the identity function. Hints:
First prove $\forall n. n \leq fn$. Then prove $f$ is monotonic. Then prove $\forall n. fn \leq n$.
§ We first prove $\forall n. n \leq fn$ by induction on $n$.

Base case: $n=0 : 0 \leq f0$ because $f : \textrm{nat} \to \textrm{nat}$
Assume $\forall n. i \leq n \Rightarrow i \leq fn$ as induction hypothesis. We must now prove

$\forall n. i+1 \leq n \Rightarrow i+1 \leq fn$ which we prove by induction on $n$.

Base case: $n=0 : i+1 \leq 0 \Rightarrow i+1 \leq 0$ has a false antecedent.
Assume $n \leq fn$ as induction hypothesis. We must now prove

$n+1 \leq f(n+1)$

$\iff n < f(n+1)$ stick two terms in between

$\iff n \leq fn < ffn < f(n+1)$

Use the induction hypothesis for $n \leq fn$. Use it again for $fn \leq ffn$ with $n$ instantiated as $fn$. Use the given information $ffn < f(n+1)$ for the final piece. Now we have proven $\forall n. n \leq fn$, which means that $f$ lies on or above the diagonal. Next we prove $\forall n. fn < ffn$.

$ffn < f(n+1)$

Now we prove $\forall n. fn \leq n$.

$fn \leq n$

$\iff fn < n+1$ use part (a) with $fn$ as $i$ and $n+1$ as $j$

$\iff ffn < f(n+1)$ use the given information.

$\iff T$

THIS PROOF NEEDS TO BE MADE FULLY CALCULATIONAL.