341 Prove that the average value of

(a) n^2 as n varies over *nat*+1 according to probability 2^{-n} is 6 .

(b) n as it varies over *nat* according to probability $(5/6)^n \times 1/6$ is 5 .

After trying the question, scroll down to the solution.

(a) n^2 as n varies over $nat+1$ according to probability 2^{-n} is 6 .

§ Lemma 0:

$$\Sigma n: nat+1 \cdot 2^{-n}$$
$$= (\Sigma n: nat \cdot 2^{-n}) - 2^{-0}$$
$$= (\Sigma n: nat+1 \cdot 2^{-(n-1)}) - 1$$
$$= (\Sigma n: nat+1 \cdot 2^{-n}\times 2) - 1$$
$$= 2\times(\Sigma n: nat+1 \cdot 2^{-n}) - 1$$

Lemma 1:

$\top$ use Lemma 0

$= (\Sigma n: nat+1 \cdot 2^{-n}) = 2\times(\Sigma n: nat+1 \cdot 2^{-n}) - 1$

$= (\Sigma n: nat+1 \cdot 2^{-n}) = 1$

and therefore, as n varies over $nat+1$, 2^{-n} is a distribution.

Lemma 2:

$$\Sigma n: nat+1 \cdot n\times 2^{-n}$$
$$= \Sigma n: nat \cdot n\times 2^{-n}$$
$$= \Sigma n: nat+1 \cdot (n-1)\times 2^{-(n-1)}$$

$= 2\times(\Sigma n: nat+1 \cdot n\times 2^{-n}) - 2\times(\Sigma n: nat+1 \cdot 2^{-n})$ use Lemma 1

$= 2\times(\Sigma n: nat+1 \cdot n\times 2^{-n}) - 2$

Lemma 3:

$\top$ use Lemma 2

$= (\Sigma n: nat+1 \cdot n\times 2^{-n}) = 2\times(\Sigma n: nat+1 \cdot n\times 2^{-n}) - 2$

$= (\Sigma n: nat+1 \cdot n\times 2^{-n}) = 2$

Lemma 4:

$$\Sigma n: nat+1 \cdot n^2\times 2^{-n}$$
$$= \Sigma n: nat \cdot n^2\times 2^{-n}$$
$$= \Sigma n: nat+1 \cdot (n-1)^2\times 2^{-(n-1)}$$

$= 2\times(\Sigma n: nat+1 \cdot n^2\times 2^{-n}) - 4\times(\Sigma n: nat+1 \cdot n\times 2^{-n}) + 2\times(\Sigma n: nat+1 \cdot 2^{-n})$ Lemmas 1 and 3

$= 2\times(\Sigma n: nat+1 \cdot n^2\times 2^{-n}) - 4\times 2 + 2$

$= 2\times(\Sigma n: nat+1 \cdot n^2\times 2^{-n}) - 6$

Lemma 5:

$\top$ use Lemma 4

$= (\Sigma n: nat+1 \cdot n^2\times 2^{-n}) = 2\times(\Sigma n: nat+1 \cdot n^2\times 2^{-n}) - 6$

$= (\Sigma n: nat+1 \cdot n^2\times 2^{-n}) = 6$

The average value of n^2 as n varies over $nat+1$ according to distribution 2^{-n} is

$2^{-n'}.\ n^2$

$= \Sigma n'': nat+1 \cdot 2^{-n''}\times n''^2$ use Lemma 5

$= 6$

(b) n as it varies over nat according to probability $(5/6)^n \times 1/6$ is 5 .

§ Lemma 6:

$$\Sigma n: nat \cdot (5/6)^n$$
$$= 1 + \Sigma n: nat+1 \cdot (5/6)^n$$
$$= 1 + \Sigma n: nat \cdot (5/6)^{n+1}$$
$$= 1 + 5/6 \times \Sigma n: nat \cdot (5/6)^n$$

Lemma 7:

$\top$ use Lemma 6

$= (\Sigma n: nat \cdot (5/6)^n) = 1 + 5/6 \times (\Sigma n: nat \cdot (5/6)^n)$

$= (\Sigma n: nat \cdot (5/6)^n) = 1 / (1 - 5/6)$

$= (\Sigma n: nat \cdot (5/6)^n) = 6$

and therefore, as n varies over nat , $(5/6)^n \times 1/6$ is a distribution.

Lemma 8:

$$\Sigma n: nat \cdot (5/6)^n \times n$$

$=\quad 0 + \Sigma n\text{: } nat{+}1\cdot\ (5/6)^n \times n$

$=\quad \Sigma n\text{: } nat\cdot\ (5/6)^{n+1} \times (n+1)$

$=\quad 5/6 \times (\Sigma n\text{: } nat\cdot\ (5/6)^n \times n) \ +\ 5/6 \times (\Sigma n\text{: } nat\cdot\ (5/6)^n)$ use Lemma 7

$=\quad 5/6 \times (\Sigma n\text{: } nat\cdot\ (5/6)^n \times n) \ +\ 5/6 \times 6$

$=\quad 5/6 \times (\Sigma n\text{: } nat\cdot\ (5/6)^n \times n) \ +\ 5$

Lemma 9:

$\quad \top$ use Lemma 8

$=\quad (\Sigma n\text{: } nat\cdot\ (5/6)^n \times n) \ = \ 5/6 \times (\Sigma n\text{: } nat\cdot\ (5/6)^n \times n) + 5$

$=\quad (\Sigma n\text{: } nat\cdot\ (5/6)^n \times n) \ = \ 5 / (1 - 5/6)$

$=\quad (\Sigma n\text{: } nat\cdot\ (5/6)^n \times n) \ = \ 30$

The average value of n as it varies over nat according to distribution $(5/6)^n \times 1/6$ is

$\quad (5/6)^{n'} \times 1/6.\ \ n$

$=\quad \Sigma n''\text{: } nat\cdot\ (5/6)^{n''} \times 1/6 \times n''$

$=\quad 1/6 \times \Sigma n''\text{: } nat\cdot\ (5/6)^{n''} \times n''$ use Lemma 9

$=\quad 1/6 \times 30$

$=\quad 5$