334 Could we define the expression P **value** e with the axiom

(a) $x'=(P \textbf{ value } e) \;=\; P.\; x'=e$

(b) $x'=(P \textbf{ value } e) \;\Longrightarrow\; P.\; x'=e$

(c) $P \;\Rightarrow\; (P \textbf{ value } e)=e'$

(d) $x'=(P \textbf{ value } e) \wedge P \;\Rightarrow\; x'=e'$

After trying the question, scroll down to the solution.

§ The **value** expression axiom in Subsection 5.5.0 is

$P.\ (P \textbf{ value } e)=e$

except that $(P \textbf{ value } e)$ is not subject to double-priming in sequential composition, nor to substitution when using the Substitution Law. In one natural variable x, consider $\top \textbf{ value } x$, first under the axiom in Subsection 5.5.0, then under each axiom offered in the exercise.

$\top.\ (\top \textbf{ value } x)=x$ sequential composition
$=\ \top$

(a) $x'=(P \textbf{ value } e)\ =\ P.\ x'=e$

§ $(x'=(\top \textbf{ value } x))\ =\ (\top.\ x'=x)$ sequential composition
$=\ (x'=(\top \textbf{ value } x))\ =\ \top$ identity
$=\ x'=(\top \textbf{ value } x)$

but we should get $\top$, so (a) is stronger than the Subsection 5.5.0 axiom. It leads to inconsistency, as follows.

$(r'<5 \textbf{ value } r)=3$ one-point law in reverse
$=\ \forall x'\cdot\ x'=(r'<5 \textbf{ value } r)\ \Rightarrow\ x'=3$ use the (a) axiom
$=\ \forall x'\cdot\ (r'<5.\ x'=r)\ \Rightarrow\ x'=3$
$=\ \forall x'\cdot\ x'<5\ \Rightarrow\ x'=3$
$=\ \bot$

$(r'<5 \textbf{ value } r)=3$ one-point law in reverse
$=\ \forall x'\cdot\ x'=3\ \Rightarrow\ x'=(r'<5 \textbf{ value } r)$ use the (a) axiom
$=\ \forall x'\cdot\ x'=3\ \Rightarrow\ (r'<5.\ x'=r)$
$=\ \forall x'\cdot\ x'=3\ \Rightarrow\ x'<5$
$=\ \top$

(b) $x'=(P \textbf{ value } e)\ \Longrightarrow\ P.\ x'=e$

§ $x'=(\top \textbf{ value } x)\ \Rightarrow\ (\top.\ x'=x)$ sequential composition
$=\ x'=(\top \textbf{ value } x)\ \Rightarrow\ \top$ base
$=\ \top$

so maybe this one is all right.

(c) $P\ \Rightarrow\ (P \textbf{ value } e)=e'$

§ $\top\ \Rightarrow\ (\top \textbf{ value } x)=x'$ identity
$=\ (\top \textbf{ value } x)=x'$

but we should get $\top$, so (c) is stronger than the Subsection 5.5.0 axiom. It leads to inconsistency, as follows.

$(r'<5 \textbf{ value } r)=3$ one-point law in reverse
$=\ \forall x'\cdot\ x'=(r'<5 \textbf{ value } r)\ \Rightarrow\ x'=3$ use the (c) axiom
$\Longrightarrow\ \forall x'\cdot\ r'<5\ \Rightarrow\ x'=3$
$=\ \neg\ r'<5$

$(r'<5 \textbf{ value } r)=3$ one-point law in reverse
$=\ \forall x'\cdot\ x'=3\ \Rightarrow\ x'=(r'<5 \textbf{ value } r)$ use the (c) axiom
$\Longleftarrow\ \forall x'\cdot\ x'=3\ \Rightarrow\ r'<5$
$=\ r'<5$

Now we have

$r'<5\ \Rightarrow\ (r'<5 \textbf{ value } r)=3\ \Rightarrow\ \neg\ r'<5$

from which we conclude $\neg\ r'<5$. By a very similar calculation, we can prove $\neg\ r'\ge 5$, contradicting trichotomy.

(d) $x'=(P \textbf{ value } e) \wedge P\ \Rightarrow\ x'=e'$

§ $x'=(\top \textbf{ value } x) \wedge \top \Rightarrow x'=x$ identity

$= \quad x'=(\top \textbf{ value } x) \Rightarrow x'=x$

but we should get $\top$, so (d) is stronger than the Subsection 5.5.0 axiom.