Could we define the programmed expression $P \text{ result } e$ with the axiom $P. (P\text{ result } e)=e$

The result expression axiom in Subsection 5.5.0 is $P. (P\text{ result } e)=e$

except that $(P\text{ result } e)$ is not subject to double-priming in sequential composition, nor to substitution when using the Substitution Law. In one natural variable $x$, consider $\top \text{ result } x$, first under the axiom in Subsection 5.5.0, then under each axiom offered in the exercise.

$$\top \text{ result } x = x$$

sequential composition

\[(a) \quad x'=(P\text{ result } e) \implies P. x'=e\]

\[= \quad (x'=(\top \text{ result } x)) \implies (\top. x'=x)\]

sequential composition

\[= \quad (x'=(\top \text{ result } x)) \implies \top\]

identity

\[= \quad x'=(\top \text{ result } x)\]

but we should get \(\top\), so (a) is stronger than the Subsection 5.5.0 axiom. It leads to inconsistency, as follows.

\[(r'<5 \text{ result } r)=3\]

one-point law in reverse

\[\equiv (\forall x'. \ x'=(r'<5 \text{ result } r) \implies x'=3)\]

use the (a) axiom

\[= \quad \forall x'. \ x'=3 \implies x'=(r'<5 \text{ result } r)\]

\[= \quad \forall x'. \ x'=3 \implies (r'<5. x'=r)\]

\[= \quad \forall x'. \ x'=3 \implies x'<5\]

\[= \quad \top\]

\[= \quad \top\]

identity

\[(b) \quad x'=(P\text{ result } e) \implies P. x'=e\]

\[= \quad x'=(\top \text{ result } x) \implies (\top. x'=x)\]

sequential composition

\[= \quad x'=(\top \text{ result } x) \implies \top\]

base

\[= \quad \top\]

so maybe this one is all right.

\[(c) \quad P \implies (P\text{ result } e)=e'\]

\[= \quad (\top \text{ result } x)=x'\]

identity

but we should get \(\top\), so (c) is stronger than the Subsection 5.5.0 axiom. It leads to inconsistency, as follows.

\[(r'<5 \text{ result } r)=3\]

one-point law in reverse

\[\equiv (\forall x'. \ x'=(r'<5 \text{ result } r) \implies x'=3)\]

use the (c) axiom

\[= \quad \forall x'. \ r'<5 \implies x'=3\]

\[= \quad \neg r'<5\]

\[(r'<5 \text{ result } r)=3\]

one-point law in reverse

\[\equiv (\forall x'. \ x'=3 \implies x'=(r'<5 \text{ result } r))\]

use the (c) axiom

\[= \quad r'<5\]

Now we have $r'<5 \implies (r'<5 \text{ result } r)=3 \implies \neg r'<5$.

from which we conclude $\neg r'<5$. By a very similar calculation, we can prove $\neg r' \geq 5$, contradicting trichotomy.
(d) \[ x' = (P \text{ result } e) \land P \Rightarrow x' = e' \]

\[ x'(\top \text{ result } x) \land \top \Rightarrow x' = x \]

\[ = x'(\top \text{ result } x) \Rightarrow x' = x \]

but we should get \( \top \), so (d) is stronger than the Subsection 5.5.0 axiom.