324 (drunk) A drunkard is trying to walk home. At each time unit, the drunkard may go forward one distance unit, stay in the same position, or go back one distance unit. After \( n \) time units, where is the drunkard?

(a) At each time unit, there is a 2/3 probability of going forward, and a 1/3 probability of staying in the same position. The drunkard does not go back.

§ The drunkard's walk can be described by the following program: Let \( x \) and \( n \) be natural variables; \( x \) describes the drunkard's position, and \( n \) the remaining time.

\[
x' \leq n \quad \iff \quad x := 0 \quad 0 \leq x' - x \leq n
\]

\[
0 \leq x' - x \leq n \quad \iff \quad \begin{cases} \text{if } n = 0 \text{ then } \text{ok} \\
\text{else if } \text{rand } 3 < 2 \text{ then } x := x + 1 \text{ else } \text{ok fi.}
\end{cases}
\]

The first occurrence of \text{ok} can be weakened to \( x' = x \), and we do so in what follows. We replace \text{rand } 3 \text{ with } \text{r: nat} \rightarrow (0,..,3) \text{ with } r \text{ having probability } 1/3.

\[
x' \leq n \quad \iff \quad x := 0 \quad 0 \leq x' - x \leq n
\]

\[
0 \leq x' - x \leq n \quad \iff \quad \begin{cases} \text{if } n = 0 \text{ then } x' = x \\
\text{else if } r < 2 \text{ then } x := x + 1 \text{ else } \text{ok fi.}
\end{cases}
\]

The first implication is proven by replacing \( x \) with 0 in \( 0 \leq x' - x \leq n \). For the second implication, in its antecedent (right side), we distribute the last line over the previous \text{then} and \text{else} parts, and write the \text{ifs} as a disjunction of conjunctions. Thus we have three cases to prove:

\[
0 \leq x' - x \leq n \iff n = 0 \land x' = x
\]

\[
0 \leq x' - x \leq n \iff n = 0 \land r < 2 \land (x := x + 1. \ n := n - 1. \ 0 \leq x' - x \leq n)
\]

\[
0 \leq x' - x \leq n \iff n = 0 \land r \geq 2 \land (\text{ok. } n := n - 1. \ 0 \leq x' - x \leq n)
\]

The first case is easy. In the middle case, start with the antecedent:

\[
\begin{align*}
n = 0 & \land r < 2 \land (x := x + 1. \ n := n - 1. \ 0 \leq x' - x \leq n) \\
\Rightarrow & \quad \begin{cases} \text{two uses of substitution} \\
\text{simplify} \\
\text{specialize, weaken}
\end{cases}
\end{align*}
\]

In the last case, start with the antecedent:

\[
\begin{align*}
n = 0 & \land r \geq 2 \land (\text{ok. } n := n - 1. \ 0 \leq x' - x \leq n) \\
\Rightarrow & \quad \begin{cases} \text{substitution and identity} \\
\text{specialize, weaken}
\end{cases}
\end{align*}
\]

So far we have proven that the computation satisfies \( x' \leq n \). A much better answer is the probability distribution of \( x' \), which will depend on \( n \). Let \( n! \) (\( n \) factorial) be defined in the usual way:

\[
n! = \prod_{i=1}^{n} i = 1! \times 2! \times 3! \times \ldots \times n!
\]

Of the \( n \) attempts to step forward, the drunkard succeeds \( x' \) times. Out of \( n \) attempts there are \( n!/(x'!(n-x')!) \) ways to choose \( x' \) attempts that succeed, each way having probability \((2/3)^x \times (1/3)^{n-x} \). So we hypothesize that after \( n \) attempts, the drunkard walks \( x' \) steps forward with probability \( n!/(x'!(n-x')!) \times 2^{x}/3^{n} \). That is the probability for specification \( x' \leq n \); the probability for the other specification \( 0 \leq x' - x \leq n \) is

\[
n!/(x'!(n-x'+x)!) \times 2^{x'-x}/3^{n}
\]

Here's the proof: start with the implementation (right side) of the first refinement.

\[
x := 0. \ (0 \leq x' - x \leq n) \times n!/(x'!(n-x')!) \times 2^{x'-x}/3^{n}
\]

which is the specification (left side) of the first refinement. Now the last refinement, starting with its implementation, replacing \text{rand} \( 3 < 2 \) with \( 2/3 \).

\[
\begin{cases} \text{if } n = 0 \text{ then } x' = x \\
\text{else if } 2/3 \text{ then } x := x + 1 \text{ else } \text{ok fi.}
\end{cases}
\]

\[
n := n - 1. \ (0 \leq x' - x \leq n) \times n!/(x'!(n-x')!) \times 2^{x'-x}/3^{n} \text{ fi}
\]
At each time unit, there is a \( \frac{1}{4} \) probability of going forward, a \( \frac{1}{2} \) probability of staying in the same position, and a \( \frac{1}{4} \) probability of going back.

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