The notation \texttt{do }P\texttt{ while }b\texttt{ od} has been used as a loop construct that is executed as follows. First, \( P \) is executed; then \( b \) is evaluated, and if its value is \( \top \) then execution is repeated, and if its value is \( \bot \) then execution is finished.

(a) Let \( x \) be an integer variable. Prove \( \mod x' 2 = \mod x 2 \iff \texttt{do }x := x-2\texttt{ while }x \geq 2\texttt{ od} \)

(b) Let \( m \) and \( n \) be integer variables. Prove \( m := m + n - 10.\ n := 10 \iff \texttt{do }m := m-1\texttt{ while }n \neq 10\texttt{ od} \)

(c) In parts (a) and (b), add a time variable, and charge time \( 1 \) for each loop iteration. Notice that for this loop, recursive time is not quite the same as charging time \( 1 \) for each iteration. Choose a time specification, and prove it.

After trying the question, scroll down to the solution.
§ In part (a), to count iterations, put the time increment as follows:
\[
\text{do } t := t + 1 \text{ while } x \geq 2 \text{ od}
\]

§ To prove \( S \) is refined by \( \text{do } P \text{ while } b \text{ od} \), prove instead
\[
S \iff P. \text{ if } b \text{ then } S \text{ else } \top \text{ fi}
\]
So we prove
\[
(mod \ x' \ 2 = mod \ x \ 2 \iff x := x-2. \text{ if } x \geq 2 \text{ then } mod \ x' \ 2 = mod \ x \ 2 \text{ else } \top \text{ fi})
\]


\[
= (mod \ x' \ 2 = mod \ x \ 2 \iff x := x-2. \text{ if } x \geq 2 \text{ then } mod \ x' \ 2 = mod \ x \ 2 \text{ else } x' = x \text{ fi})
\]

\[
= mod \ x' \ 2 = mod \ x \ 2 \iff \text{if } x-2 \geq 2 \text{ then } mod \ x' \ 2 = mod \ (x-2) \ 2 \text{ else } x' = x-2 \text{ fi}
\]

\[
= (mod \ x' \ 2 = mod \ x \ 2 \iff \text{mod } x' \ 2 = \text{mod } (x-2) \ 2)
\]

\[
\iff (mod \ x' \ 2 = mod \ x \ 2 \iff \text{mod } x' \ 2 = \text{mod } (x-2) \ 2)
\]

\[
\iff \text{context again}
\]

\[
\iff \text{specialization and specialization again}
\]

\[
\iff \text{context and context again}
\]

\[
\iff \text{substitution law twice}
\]

\[
\iff \text{if and then parts arithmetic; in else part context: } n=9
\]

\[
\iff \text{case idempotent}
\]

\[
\iff \text{definition of assignment and sequential composition}
\]

\[
\iff m := m+n-10. \text{ } n := 10
\]

(b) Let \( m \) and \( n \) be integer variables. Prove
\[
m:= m+n-10. \text{ } n := 10 \iff \text{do } m := m-1. \text{ } n := n+1 \text{ while } n \neq 10 \text{ od}
\]

§ Apparently, we are not talking about time in this question; we don't have variable \( t \). So we can't talk about termination or nontermination, because those are timing issues.

I prove
\[
m := m+n-10. \text{ } n := 10 \iff \text{if } n \neq 10 \text{ then } m := m+n-10. \text{ } n := 10 \text{ else } \top \text{ fi}
\]
starting with the right side.
\[
m := m-1. \text{ } n := n+1. \text{ if } n \neq 10 \text{ then } m := m+n-10. \text{ } n := 10 \text{ else } \top \text{ fi}
\]
replace \( n := 10 \) and \( \top \).
\[
m := m-1. \text{ } n := n+1. \text{ if } n \neq 10 \text{ then } m := m+n-10. \text{ } m' = m \land n' = 10 \text{ else } m' = m \land n' = n \text{ fi}
\]
substitution law in then part.
\[
m := m-1. \text{ } n := n+1. \text{ if } n \neq 10 \text{ then } m' := m+n-10 \land n' = 10 \text{ else } m' = m \land n' = n \text{ fi}
\]
substitution law twice.
\[
\text{if } n+1 \neq 10 \text{ then } m' = m-1+n+1-10 \land n' = 10 \text{ else } m' = m-1 \land n' = n+1 \text{ fi}
\]
if and then parts arithmetic; in else part context: \( n=9 \).
\[
\text{if } n+9 \text{ then } m' = m+n-10 \land n' = 10 \text{ else } m' = m+n-10 \land n' = n \text{ fi}
\]
case idempotent.
\[
\iff m' = m+n-10 \land n' = 10
\]
definition of assignment and sequential composition.
\[
\iff m := m+n-10. \text{ } n := 10
\]

(c) In parts (a) and (b), add a time variable, and charge time 1 for each loop iteration. Notice that for this loop, recursive time is not quite the same as charging time 1 for each iteration. Choose a time specification, and prove it.

§ In part (a), to count iterations, put the time increment as follows:
\[
\text{do } t := t+1. \text{ } x := x-2 \text{ while } x \geq 2 \text{ od}
\]
My time specification is
\[
\text{if } x \geq 2 \text{ then } t' = t + \text{floor}(x/2) \text{ else } t' = t+1 \text{ fi}
\]
But \( \text{floor} \) is an awkward function to deal with, so I weaken my specification slightly to
\[
\text{if } x \geq 2 \text{ then } t' \leq t + x/2 \text{ else } t' = t+1 \text{ fi}
\]
So I prove
\[
\text{if } x \geq 2 \text{ then } t' \leq t + x/2 \text{ else } t' = t+1 \text{ fi}
\]

\[
\text{starting with the right (bottom) side:}
\]
\[
 t := t+1. \ x := x-2. \text{ if } x \geq 2 \text{ then if } x \geq 2 \text{ then } t' \leq t + x/2 \text{ else } t' = t+1 \text{ fi else ok fi}
\]

\[
\text{context } x \geq 2
\]

\[
\Rightarrow t := t+1. \ x := x-2. \text{ if } x \geq 2 \text{ then } t' \leq t + x/2 \text{ else ok fi}
\]

\[
\Rightarrow t := t+1. \ x := x-2. \text{ if } x \geq 2 \text{ then } t' \leq t + x/2 \text{ else } t' = t+1 \text{ fi}
\]

\[
\Rightarrow \text{ if } x \geq 4 \text{ then } t' \leq t + x/2 \text{ else } t' = t+1 \text{ fi}
\]

\[
\text{In part (b), to count iterations, put the time increment as follows:}
\]
\[
\text{do } t := t+1. \ m := m-1. \ n := n+1 \text{ while } n \neq 10 \text{ od}
\]

\[
\text{My time specification is if } n < 10 \text{ then } t' = t+10-n \text{ else } t' = \infty \text{ fi}
\]

\[
\text{So I prove}
\]
\[
\text{if } n < 10 \text{ then } t' = t+10-n \text{ else } t' = \infty \text{ fi}
\]

\[
\text{starting with the right (bottom) side.}
\]
\[
 t := t+1. \ m := m-1. \ n := n+1.
\]
\[
\text{if } n + 10 \text{ then if } n < 10 \text{ then } t' = t+10-n \text{ else } t' = \infty \text{ else ok fi}
\]

\[
\Rightarrow t := t+1. \ m := m-1. \ n := n+1.
\]
\[
\text{if } n + 10 \text{ then if } n < 10 \text{ then } t' = t+10-n \text{ else } t' = \infty \text{ else } t' = t \text{ fi}
\]

\[
\Rightarrow \text{ if } n + 10 \text{ then if } n < 10 \text{ then } t' = t+10-n \text{ else } t' = \infty \text{ fi else } t' = t \text{ fi}
\]

\[
\Rightarrow \text{ if } n + 10 \text{ then if } n < 10 \text{ then } t' = t+10-n \text{ else } t' = \infty \text{ fi else } t' = t+1 \text{ fi}
\]

\[
\Rightarrow \text{ if } n + 10 \text{ then if } n < 10 \text{ then } t' = t+10-n \text{ else } t' = \infty \text{ fi else } t' = t+1 \text{ fi}
\]

\[
\Rightarrow \text{ if } n + 10 \text{ then if } n < 10 \text{ then } t' = t+10-n \text{ else } t' = \infty \text{ fi else } t' = t+1 \text{ fi}
\]

\[
\Rightarrow \text{ if } n + 9 \text{ then if } n < 9 \text{ then } t' = t+10-n \text{ else } t' = \infty \text{ fi else } t' = t+1 \text{ fi}
\]

\[
\Rightarrow \text{ if } n + 9 \text{ then if } n < 9 \text{ then } t' = t+10-n \text{ else } t' = \infty \text{ fi else } t' = t+10-n \text{ fi}
\]

\[
\Rightarrow \text{ if } n < 10 \text{ then } t' = t+10-n \text{ else } t' = \infty \text{ fi}
\]