For what exact precondition and postcondition does the following assignment move integer variable \( x \) farther from zero staying on the same side of zero?

§ What does “staying on the same side of zero” mean if the initial value of \( x \) is zero? Since that's not clear, let’s say that \( x' \) can be on either side in that case. The specification is:

\[
(x < 0 \Rightarrow x' < 0) \land (x = 0 \Rightarrow x' > 0) \land (x > 0 \Rightarrow x' > 0)
\]

\[
= x' < 0 \lor x' + x = 0 \lor x' > 0
\]

(a) \( x := x + 1 \)

§ (exact precondition for \( x' < x < 0 \lor x' + x = 0 \lor x' > x > 0 \) to be refined by \( x := x + 1 \))

\[
\forall x'. \quad x' < x < 0 \lor x' + x = 0 \lor x' > x > 0 \iff (x := x + 1)
\]

\[
\forall x'. \quad x' < x < 0 \lor x' + x = 0 \lor x' > x > 0 \iff x' = x + 1 \quad \text{One-point}
\]

\[
x + 1 < x < 0 \lor x + 1 + x = 0 \lor x + 1 > x > 0
\]

\[
\perp \lor x = 0 \lor x > 0
\]

\[
x = 0
\]

We can be sure that \( x := x + 1 \) will move \( x \) farther from zero, staying on the same side, if \( x \geq 0 \).

(b) \( x := \text{abs} \ (x + 1) \)

§ (exact precondition for \( x' < x < 0 \lor x' + x = 0 \lor x' > x > 0 \) to be refined by \( x := \text{abs} \ (x + 1) \))

\[
\forall x'. \quad x' < x < 0 \lor x' + x = 0 \lor x' > x > 0 \iff (x := \text{abs} \ (x + 1))
\]

\[
\forall x'. \quad x' < x < 0 \lor x' + x = 0 \lor x' > x > 0 \iff x' = \text{abs} \ (x + 1) \quad \text{One-point}
\]

\[
\text{abs} \ (x + 1) < x < 0 \lor \text{abs} \ (x + 1) + x = 0 \lor \text{abs} \ (x + 1) > x > 0
\]

\[
\perp \lor x = 0 \lor x > 0
\]

\[
x = 0
\]

We can be sure that \( x := \text{abs} \ (x + 1) \) will move \( x \) farther from zero, staying on the same side, if \( x \geq 0 \).

(c) \( x := x^2 \)

§ (exact precondition for \( x' < x < 0 \lor x' + x = 0 \lor x' > x > 0 \) to be refined by \( x := x^2 \))

\[
\forall x'. \quad x' < x < 0 \lor x' + x = 0 \lor x' > x > 0 \iff (x := x^2)
\]

\[
\forall x'. \quad x' < x < 0 \lor x' + x = 0 \lor x' > x > 0 \iff x' = x^2 \quad \text{One-point}
\]
\( x^2 < 0 \) \lor \ x^2 + x = 0 \lor \ x^2 > 0 \)
\( x \geq 2 \)
We can be sure that \( x := x^2 \) will move \( x \) farther from zero, staying on the same side, if \( x \geq 2 \).

(exact postcondition for \( x' < 0 \lor x' + x = 0 \lor x' > 0 \) to be refined by \( x := x^2 \))
\( \forall x : x' < 0 \lor x' + x = 0 \lor x' > 0 \iff (x := x^2) \)
\( \forall x : x^2 < 0 \lor x^2 + x = 0 \lor x^2 > 0 \iff x' = x^2 \)
\( \forall x : \bot \lor \bot \lor x > 1 \iff x' = x^2 \)
contrapositive
\( \forall x : x \leq 1 \Rightarrow x' + x^2 \)
\( \forall x : x \leq 1 \Rightarrow x' + x^2 \)
\( (\forall x : x \leq 1 \Rightarrow x' + x^2) \land (\forall x : x \leq 1 \Rightarrow x' + x^2) \)
\( (\forall x : x \leq 1 \Rightarrow x' + x^2) \land (\forall x : -x \leq 1 \Rightarrow x' + (-x)^2) \)
\( (\forall x : x \leq 1 \Rightarrow x' + x^2) \land (\forall x : x > -1 \Rightarrow x' + x^2) \)
combine
\( \forall x : x' + x^2 \)
We can be sure that \( x := x^2 \) moved \( x \) farther from zero, staying on the same side, if \( x' \) is not a square. But of course it will be a square, so we can never be sure that \( x \) moved farther from zero, staying on the same side.