For what exact precondition and exact postcondition does the following assignment move integer variable $x$ farther from zero?

(a) \( x := x + 1 \)
(b) \( x := \text{abs} \ (x+1) \)
(c) \( x := x^2 \)

After trying the question, scroll down to the solution.
(a) \(x := x + 1\)

§

(the exact precondition for \(abs' > abs\) to be refined by \(x := x + 1\))

\[\forall x' \cdot abs' > abs \iff (x := x + 1)\]
\[\forall x' \cdot abs' > abs \iff x' = x + 1\]
\[abs(x + 1) > abs x\]
\[x \geq 0\]

(the exact postcondition for \(abs' > abs\) to be refined by \(x := x + 1\))

\[\forall x \cdot abs' > abs \iff (x := x + 1)\]
\[\forall x \cdot abs' > abs \iff x' = abs(x + 1)\]
\[abs' > abs(x' - 1)\]
\[x' \geq 1\]

(b) \(x := abs\ (x + 1)\)

§

(the exact precondition for \(abs' > abs\) to be refined by \(x := abs\ (x + 1)\))

\[\forall x' \cdot abs' > abs \iff (x := abs(x + 1))\]
\[\forall x' \cdot abs' > abs \iff x' = abs(x + 1)\]
\[abs\ (abs(x + 1)) > abs x\]
\[x \geq 0\]

(the exact postcondition for \(abs' > abs\) to be refined by \(x := abs\ (x + 1)\))

\[\forall x \cdot abs' > abs \iff (x := abs(x + 1))\]
\[\forall x \cdot abs' > abs \iff x' = abs(x + 1)\]
\[abs(x + 1) > abs x\]
\[\forall x \cdot nat \cdot abs' > abs x \iff x' = abs(x + 1)\]
\[\forall x \cdot nat \cdot abs' > abs x \iff x' = abs(x + 1)\]
\[\forall z \cdot nat \cdot abs' > abs (-z - 1) \iff x' = abs(-z - 1 + 1)\]
\[\forall x \cdot nat \cdot abs' > abs x \iff x' = abs(x + 1)\]
\[\forall x \cdot nat \cdot abs' > abs x \iff x' = abs(x + 1)\]
\[\forall z \cdot nat \cdot abs' > abs + 1 \iff x' = z\]
\[\forall x \cdot nat \cdot abs (x + 1) > x \iff x' = x + 1\]
\[\forall x \cdot nat \cdot abs (x + 1) > x \iff x' = x + 1\]
\[\forall z \cdot nat \cdot abs > z + 1 \iff x' = z\]
\[\forall x \cdot nat \cdot abs > z + 1 \iff x' = z\]
\[\forall z \cdot nat \cdot abs > z + 1 \iff x' = z\]
\[\forall z \cdot nat \cdot abs + 1 \iff x' = z\]
\[\forall x \cdot nat \cdot T \iff x' = x + 1\]
\[\forall z \cdot nat \cdot T \iff x' = z\]
\[\forall z \cdot nat \cdot \bot \iff x' = z\]
\[\forall x \cdot nat \cdot x' = z\]
\[\forall x \cdot nat \cdot x' = z\]
\[\forall x \cdot nat \cdot x' = z\]
\[\forall y \cdot nat \cdot y' = y\]
\[\forall y \cdot nat \cdot y > y \iff x' = y^2\]

(c) \(x := x^2\)

§

(the exact precondition for \(abs' > abs\) to be refined by \(x := x^2\))

\[\forall x' \cdot abs' > abs \iff x' = x^2\]

One-Point Law

by the arithmetic properties of \(abs\) and \(x^2\)

\[x^2 > abs x \iff x^2 = (abs x)^2\]
\[x^2 > abs x \iff x^2 = (abs x)^2\]
\[x + 1 \land x > 0 \land x + 1\]

(the exact postcondition for \(abs' > abs\) to be refined by \(x := x^2\))

\[\forall x \cdot int \cdot abs' > abs \iff x' = x^2\]

arithmetic: \(x^2 = (-x)^2\)
\[\forall x \cdot int \cdot abs' > abs \iff x' = (abs x)^2\]

arithmetic: \(y^2 \geq 0\)
\[\forall y \cdot nat \cdot abs (y^2) > y \iff x' = y^2\]

change variable
\[\forall y \cdot nat \cdot y > y \iff x' = y^2\]

context
\[\forall y \cdot nat \cdot y \geq y\]

domain split
\[\begin{align*}
&= (\forall y: 0 \cdot y^2 > y \iff x' = y^2) \land (\forall y: 1 \cdot y^2 > y \iff x' = y^2) \\
&\quad \land (\forall y: \text{nat}+2 \cdot y^2 > y \iff x' = y^2) \\
&= (\bot \iff x' = 0) \land (\bot \iff x' = 1) \land (\forall y: \text{nat}+2 \cdot \top \iff x' = y^2) \\
&= x' \neq 0 \land x' \neq 1
\end{align*}\]