30 (hats) There are three people, named Front, Middle, and Back, standing in a queue. Each is wearing a hat, which is either red or blue. Back can see Middle's hat and Front's hat, but cannot see Back's hat. Middle can see Front's hat, but cannot see Back's hat or Middle's hat. Front cannot see anyone's hat. A passerby tells them that at least one of them is wearing a red hat. Back then says: I don't know what color hat I'm wearing. Middle then says: I don't know what color hat I'm wearing. Front then says: I do know what color hat I'm wearing. Formalize Front's reasoning, and determine what color hat Front is wearing.

After trying the question, scroll down to the solution.

§ Let b mean "Back is wearing red", let m mean "Middle is wearing red", and let f mean "Front is wearing red". The passerby said

(a) $b \lor m \lor f$

Here is Front's reasoning. If Back saw that Middle and Front were wearing blue, then Back would have calculated

$$(b \lor m \lor f) \land \neg m \land \neg f \qquad \text{duality}$$
$$= (b \lor m \lor f) \land \neg(m \lor f) \qquad \text{law of direct proof}$$
$$\Rightarrow b$$

so Back would have known that Back is wearing a red hat. So Middle and Front conclude

(b) $m \lor f$

Now if Middle saw that Front were wearing blue, Middle would have calculated

$$(m \lor f) \land \neg f \qquad \text{law of direct proof}$$
$$\Rightarrow m$$

so Middle would have known that Middle is wearing a red hat. So Front concludes f. Front is wearing red.

The puzzle could be told with more than three people, or with fewer. It can even be told about one person, but then it is just the following: One person doesn't know what color hat they are wearing. A passerby tells them it's red. So they concludes it's red. (I suppose the story can even be told about zero people: A passerby says "true". End of story.)