We can compute $x := n!$ (factorial) as follows.

$$x := n! \iff \text{if } n = 0 \text{ then } x := 1 \text{ else } n := n-1. \ x := n! \ . \ n := n+1. \ x := x \cdot n \ \text{fi}$$

Each call $x := n!$ pushes a return address onto a stack, and each return pops an address from the stack. Add a space variable $s$ and a maximum space variable $m$, with appropriate assignments to them in the program. Find and prove an upper bound on the maximum space used.

## Case Study

$m \geq s \implies m' = m \uparrow (s+n) \iff$

- if $n = 0$ then $x := 1$
- else $n := n-1.$

$s := s+1. \ m := m \uparrow s. \ m \geq s \implies m' = m \uparrow (s+n). \ s := s-1.$

$n := n+1. \ x := x \cdot n \ \text{fi}$

**Proof:** by cases. First case:

$\Rightarrow (n=0 \wedge (x:=1) \implies (m \geq s \implies m' = m \uparrow (s+n)))$

context

$\Rightarrow n=0 \wedge x' = 1 \wedge n' = n \wedge s' = s \wedge m' = m \wedge m \geq s \implies m' = m \uparrow (s+n)$

base

$\Rightarrow \top$

Second case:

$(m \geq s \implies m' = m \uparrow (s+n)) \iff$

$\Rightarrow (n+0 \wedge (n:=n-1. \ s:=s+1.\
\quad m := m \uparrow s. \ m \geq s \implies m' = m \uparrow (s+n). \ s := s-1.\
\quad n := n+1. \ x := x \cdot n))$

context

$\Rightarrow n+0 \wedge m \geq s$

simplify first $\uparrow$ to $\top$ and $+1-1$ to $0.$

In the context where natural $n$ is not $0$, $s + n \geq s + 1$ so remove $s+1$ from $\uparrow$.

$\Rightarrow n+0 \wedge m \geq s$

Eliminate $\cdot$ and then use one-point.

$\Rightarrow m' = m \uparrow (s+n)$

specialize

$\Rightarrow m' = m \uparrow (s+n)$